NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 1

Selected answers and solutions

1. (i) Write $f = u + iv$. Since f is analytic, by CR eqn, we have

$$u_x = v_y, \quad \& \quad u_y = -v_x.$$

Now $\bar{f} = u + i(-v)$ is also analytic. Thus by CR eqn for $\bar{f}$, we have

$$u_x = (-v)_y \quad \& \quad u_y = -(-v)_x.$$

Solving the 4 equations, we get

$$u_x = u_y = v_x = v_y \equiv 0.$$

Thus, $f'(z) = u_x + iv_x \equiv 0$. Thus f is constant in D.

(ii) Write $|f(z)| \equiv c$ for some complex constant c.
Case (a): $c = 0$. In this case, $|f(z)| \equiv 0 \implies f(z) \equiv 0$. Thus f is a constant function.
Case (b): $c > 0$. In this case, $|f(z)| \equiv c > 0$. So $f(z)$ is nowhere zero in D. Then $f(z)\overline{f(z)} = |f(z)|^2 \equiv c^2 \implies \overline{f(z)} = \dfrac{c^2}{f(z)}$, and thus $\overline{f(z)}$ is also an analytic function on D. By (i), then $f(z)$ is a constant function on D.

2. Let f be analytic in the closed disk $|z| \leq r$, where $r > 0$. Suppose $|f(z)| \leq M$ for all $|z| \leq r$, where $M > 0$ (sometimes, we simply say f is bounded by M on the circle $|z| = r$). Show that for any integer $n \geq 1$,

$$|f^{(n)}(z)| \leq \frac{n!M}{(r - |z|)^n} \quad \text{for all } |z| < r.$$

[Note: We may weaken the condition "$|f(z)| \leq M$ for all $|z| \leq r$" to "$|f(z)| \leq M$ for all $|z| = r$" (the MMP implies that the two inequalities are actually equivalent, why?)].

Solution: For any z with $|z| < r$, consider the circle C centered at z and of radius $r - |z|$. By the Cauchy integral formula for derivatives,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s - z)^{n+1}} dz.$$

Next we apply the ML inequality. Clearly by plane geometry, $L = 2\pi(r - |z|)$.

For any $s \in C$,

$$\left| \frac{f(s)}{(s-z)^{n+1}} \right| = \frac{|f(s)|}{|s-z|^{n+1}} \le \frac{M}{(r-|z|)^{n+1}}.$$

Thus, by the ML inequality, we have

$$|f^{(n)}(z)| \le \frac{n!}{2\pi} \cdot \frac{M}{(r-|z|)^{n+1}} \cdot 2\pi(r-|z|) = \frac{n!M}{(r-|z|)^{n}}.$$

(You should draw a diagram to illustrate your answer).

3. If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial such that $|p(z)| \le 1$ for all z satisfying $|z| \le 1$, show that $|a_k| \le 1$ for each $k = 0, 1, \cdots, n$.

   Solution: By Taylor's theorem or by direct differentiation, one easily sees that

$$a_k = \frac{p^{(k)}(0)}{k!}.$$

   Then apply the result of Question 2 (with $M = 1$ and $r = 1$ and $z = 0$). Alternatively, one may use the CIF for derivatives and the ML inequality directly to get the desired inequality.

4. Let $f(z)$ be an analytic function which has a zero of order m at z_o. (Recall that an analytic function $f(z)$ is said to have **a zero of order** m at z_o if

$$f(z_o) = f'(z_o) = f''(z_o) = \cdots = f^{(m-1)}(z_o) = 0, \quad \text{but } f^{(m)}(z_o) \ne 0.)$$

   (i) Show that there exists $r > 0$ and an analytic function $\phi(z)$ on the open ball $|z - z_o| < r$ such that $\phi(z_o) \ne 0$ and

$$f(z) = (z - z_o)^m \phi(z) \quad \text{for all } |z - z_o| < r.$$

   (ii) Show that $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_o, and show that the residue of $\dfrac{f'(z)}{f(z)}$ at z_o is equal to m.

   Solution: For part (i), use the Taylor series expansion of $f(z)$ about z_0.

   For part (ii), we recall from part (i) that

$$f(z) = (z - z_o)^m \phi(z) \quad \text{for all } |z - z_o| < r,$$

where $\phi(z)$ is analytic at z_o and $\phi(z_o) \neq 0$. Then by the Product Rule, we have, for $0 < |z - z_o| < r$,

$$\begin{aligned} f'(z) &= m(z-z_o)^{m-1}\phi(z) + (z-z_o)^m \phi'(z) \\ \Longrightarrow \quad \frac{f'(z)}{f(z)} &= \frac{m(z-z_o)^{m-1}\phi(z) + (z-z_o)^m\phi'(z)}{(z-z_o)^m\phi(z)} \\ &= \frac{m}{z-z_o} + \frac{\phi'(z)}{\phi(z)} \\ &= \frac{m}{z-z_o} + \sum_{k=0}^{\infty} a_k (z-z_o)^k, \end{aligned} \qquad (*)$$

where $\sum_{k=0}^{\infty} a_k(z-z_o)^k$ is the Taylor series of $\dfrac{\phi'(z)}{\phi(z)}$ at z_o (why is $\dfrac{\phi'(z)}{\phi(z)}$ analytic at z_o?). On the other hand, (*) is the Laurent series of $\dfrac{f'(z)}{f(z)}$, and from (*), one easily sees that $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_o, and its residue at z_o is m.

5. Let $f(z)$ and $g(z)$ be two entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there is a constant c such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$. Justify your arguments carefully.

   [Remark: $g(z)$ may be equal to zero at some points. So $f(z)/g(z)$ is apriori not an entire function.]
   Solution: First note that the zeros of g are isolated (unless $g(z) \equiv 0$ in which case the equality $f(z) \equiv cg(z)$ holds trivially), and singularities of $\frac{f}{g}$ can occur only at the zeros of g, so they are isolated. Furthermore, if z_0 is a zero of g, by considering the power series of f and g about z_0, we see that f/g has a pole if the order of the zero of g at z_0 is greater than the order of the zero of f at z_0, and a removable singularity if the order of the zero of g is less than or equal to the order of the zero of f at z_0 (result from MA3111). If f/g has a pole at z_0, then $\lim_{z\to z_0} |\frac{f}{g}(z)| = \infty$ which contradicts $|f(z)| \leq |g(z)|$ for all z. Hence all the singularities of f/g are removable and we can extend f/g to an entire function (see MA4247 lecture notes, Part 1, Page 23). Furthermore, by continuity, since $|\frac{f}{g}(z)| \leq 1$ away from the removable singularities, $|\frac{f}{g}(z)| \leq 1$ for the extended entire function. By Liouville's theorem, $\frac{f}{g}(z) = c$ for some constant c, hence $f(z) = cg(z)$.

6. (i) Find the Laurent series of the function $\dfrac{5z-7}{(z-4)(z+9)}$ for the annular domain $4 < |z| < 9$.

(ii) Using part (i) or otherwise, find the Laurent series of the function

$$\frac{(z-2)^2(5z-17)}{z^2+z-42}$$

for the annular domain $4 < |z-2| < 9$.

Answer: (i) $f(z) = \sum_{n=0}^{\infty} \frac{4^n}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{4(-1)^n}{9^{n+1}} z^n, \quad 4 < |z| < 9.$

(ii)

$$\begin{aligned} g(z) &= (z-2)^2 f(z-2) \\ &= (z-2)^2 \Big[\sum_{n=0}^{\infty} \frac{4^n}{(z-2)^{n+1}} + \sum_{n=0}^{\infty} \frac{4(-1)^n}{9^{n+1}} (z-2)^n \Big], \quad 4 < |z-2| < 9 \\ &= \sum_{n=0}^{\infty} \frac{4^n}{(z-2)^{n-1}} + \sum_{n=0}^{\infty} \frac{4(-1)^n}{9^{n+1}} (z-2)^{n+2} \Big], \quad 4 < |z-2| < 9. \end{aligned}$$