NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Supplementary Exercise 2

These are some problems related to what has been covered, up to but excluding Rouche's Theorem (questions on Rouche's theorem will be given in the next supplementary exercise set) which you may like to try out, either individually, or with your group members.

1. Construct an analytic function f on the open ball $D : |z| < 1$ which has infinitely many zeroes in D such that $f(z) \not\equiv 0$ on D. Does the zeroes of such f have an accumulation point? Explain also why such f does not contradict the identity theorem for analytic functions.

2. (a) Let f be analytic in the unit ball $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ and such that
$$\lim_{|z|\to 1-} f(z) = 0 \ .$$
Prove that $f \equiv 0$ on $B(0,1)$.

   (b) Let g be analytic in $B(0,1)$. Prove that the statement
$$\lim_{|z|\to 1-} g(z) = \infty$$
is impossible.

3. Show that if $f(z)$ is analytic inside and on a simple closed contour C, then Re $f(z)$ and Im $f(z)$ attains their maximum and minimum values on the boundary C.

4. Prove that if $f(z)$ is analytic inside and on a simple closed contour C and is one-to-one on C, then $f(z)$ is one-to-one inside C. (Hint: Consider the image curve $f(C)$.)

5. Give an example of a non-constant analytic function f defined on a simply connected domain D such that $f(D)$ is not-simply connected.
   [Recall that the Open Mapping Theorem implies that if f is a non-constant analytic function defined on a domain D, then $f(D)$ is also a domain.]

List of Supplementary Exercises from Churchill's book (7th edition):

Maximum Modulus Principle: p.173, Questions 6,7,8,9

Winding Numbers, Argument Principle, Rouché's Principle: p. 285-287, Questions 1-12.