2011/2012 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

February 28, 2012

8:30pm - 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Thirteen (13)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae Sheet

1. Integrating factor for $y' + Py = Q$ is given by

$$R = \exp\left(\int P dx\right).$$

2. The variation of parameters formulae for $y'' + py' + qy = r$:

$$u = \int \frac{-ry_2}{y_1 y_2' - y_2 y_1'} dx$$

$$v = \int \frac{ry_1}{y_1 y_2' - y_2 y_1'} dx \, .$$

1. Let y be a solution of the differential equation

$$xy\frac{dy}{dx} = \frac{1}{2}$$

such that

$$x > \frac{1}{e},\ y > 0, \text{ and } y(1) = 1.$$

Then $y\left(e^8\right) =$

(A) $\ln\frac{8}{3}$

(B) $\ln 2$

(C) 8

(D) 3

(E) None of the above

2. Let $y > 0$ be a solution of the differential equation

$$(x + 2y - 1) + 3\,(x + 2y)\,\frac{dy}{dx} = 0$$

such that

$$x > 0, \text{ and } y(1) = 1.$$

If $y\,(3) = 5a$, then a satisfies the equation

(A) $15a = 1 + 3\ln(1 + 2a)$

(B) $5a = 1 + 3\ln(3 + 10a)$

(**C**) $a = 3 - \ln(3 + 10a)$

(**D**) $3a = 5 - 3\ln(1 + 2a)$

(**E**) None of the above

3. Let y be a solution of the differential equation

$$2\frac{dy}{dx} - 2y = x$$

such that

$$y(0) = \frac{1}{2}.$$

Then $y\,(1) =$

(A) $e - \frac{1}{2}$

(B) $-\frac{1}{2}$

(C) $e - 1$

(D) 1

(E) None of the above

4. Let y be a solution of the differential equation

$$\frac{dy}{dx} + 3y = y^2$$

such that

$$y(0) = 3.$$

Then $y\,(1506) =$

(A) 2012

(B) 1509

(C) 18

(D) 3

(E) None of the above

5. A party is being held in a room that contains 1800 cubic feet of air which is originally free of carbon monoxide. Beginning at time $t = 0$, several people started smoking and smoke containing 6 % carbon monoxide was introduced into the room at a rate of 0.15 cubic feet per minute. The well-circulated air mixture left the room at the same rate through a small open window. At time $t = a$ minutes, the concentration of carbon monoxide in the room was found to be 0.018 %. What is the value of a? Give your answer correct to the nearest integer.

(A) 29

(B) 36

(C) 42

(D) 47

(E) None of the above

6. Mayflies are a kind of insect which hatch from eggs. The eggs hatch at a rate proportional to the number present, with a half-life of two days. After hatching, the mayflies begin to die at a rate proportional to the number of mayflies present, with a half-life of one day. Initially at time $t = 0$, there are one million eggs and no mayflies. How many mayflies are there at time $t = 3$?

(A) $\sqrt{2} \times 10^5$

(B) $\left(2^{-\frac{3}{2}} - 2^{-3}\right) \times 10^6$

(C) $\left(2^{-\frac{1}{4}} - 2^{-\frac{1}{2}}\right) \times 10^6$

(D) $\left(2^{\frac{1}{3}} - 2^{\frac{1}{6}}\right) \times 10^6$

(E) None of the above

7. The general solution of the differential equation

$$y'' - 2y' + y = 0$$

is

(A) $y = c_1 e^x + c_2 e^{2x}$

(B) $y = c_1 e^x + c_2 e^{-x}$

(C) $y = c_1 e^x + c_2 x e^x$

(D) $y = c_1 e^{-x} + c_2 x e^{-x}$

(E) None of the above

8. Let y be a solution of the differential equation

$$y'' - 3y' + 2y = (2x - 1)\, e^{2x}$$

such that

$$y(0) = 3,\ y'(0) = 1.$$

Find the value of $y(1)$.

(A) $e(2 - e)$

(B) $e(e - 2)$

(C) $e(1 + e)$

(D) $e(1 - e)$

(E) None of the above

9. Let y be a solution of the differential equation

$$y'' + y = 2\sin x$$

such that

$$y(0) = 0, \text{ and } y'(0) = 0.$$

Find the exact value of $y(\frac{\pi}{3})$.

(A) $\frac{3\sqrt{3}-\pi}{6}$

(B) $\frac{3+\pi}{6}$

(C) $\frac{\pi-\sqrt{3}}{6}$

(D) $\frac{\pi-\sqrt{2}}{6}$

(E) None of the above

10. Let $u\cos 3x + v\sin 3x$ be a particular solution of the differential equation

$$y'' + 9y = \csc 3x$$

where $0 < x < \frac{\pi}{6}$.

Then $v =$

(A) $-\frac{1}{3}x + C$

(B) $\frac{1}{9}x + C$

(C) $\frac{1}{3}\ln(\sin 3x) + C$

(D) $\frac{1}{9}\ln(\sin 3x) + C$

(E) None of the above

END OF PAPER

Blank page for you to do your calculations

Answers to mid term test

1. D
2. A
3. C
4. D
5. B
6. B
7. C
8. A
9. A
10. D

2012 - MA1506 : Mid Term Test Solutions

1). D

$$2y\,dy = \frac{1}{x}\,dx, \quad x > \frac{1}{e},\ y > 0$$

$$y^2 = \ln x + C$$

$$y(1) = 1 \Rightarrow C = 1$$

$$\therefore y^2 = \ln x + 1$$

$$x = e^8 \Rightarrow y^2 = \ln e^8 + 1 = 9$$

$$\Rightarrow y = \underline{\underline{3}}$$

2). A

$$\text{Let } x + 2y = u \Rightarrow 1 + \frac{2dy}{dx} = \frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 1\right)$$

$$u - 1 + 3u\,\frac{1}{2}\left(\frac{du}{dx} - 1\right) = 0$$

$$2u - 2 + 3u\frac{du}{dx} - 3u = 0$$

$$3u\frac{du}{dx} = u + 2$$

$$\frac{u}{u+2}\,du = \frac{1}{3}\,dx$$

$$\left(1 - \frac{2}{u+2}\right)du = \frac{1}{3}\,dx$$

$$u - 2\ln|u+2| = \frac{1}{3}x + C$$

$$x + 2y - 2\ln|x + 2y + 2| = \frac{1}{3}x + C$$

$$y(1) = 1 \Rightarrow C = \frac{8}{3} - 2\ln 5$$

$$\therefore x + 2y - 2\ln(x + 2y + 2) = \frac{1}{3}x + \frac{8}{3} - 2\ln 5 \quad \left(\because \begin{matrix} x>0 \\ y>0 \end{matrix}\right)$$

$$y(3) = 5a \Rightarrow 3 + 10a - 2\ln(5 + 10a) = 1 + \frac{8}{3} - 2\ln 5$$

$$\Rightarrow 10a - 2\ln 5 - 2\ln(1 + 2a) = \frac{2}{3} - 2\ln 5$$

$$\Rightarrow \underline{\underline{15a = 1 + 3\ln(1 + 2a)}}$$

3) C

$$\frac{dy}{dx} - y = \frac{1}{2}x$$

$$R = e^{\int -dx} = e^{-x}$$

$$y = e^{x} \int \frac{1}{2} x e^{-x} dx$$

$$= e^{x} \int -\frac{1}{2} x \, d(e^{-x})$$

$$= e^{x} \left\{ -\frac{1}{2} x e^{-x} + \int \frac{1}{2} e^{-x} dx \right\}$$

$$= e^{x} \left\{ -\frac{1}{2} x e^{-x} - \frac{1}{2} e^{-x} + C \right\}$$

$$= -\frac{1}{2} x - \frac{1}{2} + Ce^{x}$$

$$y(0) = \frac{1}{2} \Rightarrow C = 1$$

$$y(1) = -\frac{1}{2} - \frac{1}{2} + e = \underline{\underline{e - 1}}$$

4). D

Let $z = y^{1-2} = y^{-1}$

$$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\therefore -y^2\frac{dz}{dx} + 3y = y^2$$

$$\frac{dz}{dx} - 3z = -1$$

$$R = e^{\int -3dx} = e^{-3x}$$

$$z = e^{3x}\int -e^{-3x}dx$$

$$= e^{3x}\left\{\frac{1}{3}e^{-3x} + c\right\}$$

$$\therefore \frac{1}{y} = \frac{1}{3} + ce^{3x}$$

$$y(0) = 3 \Rightarrow \frac{1}{3} = \frac{1}{3} + c \Rightarrow c = 0 \Rightarrow y \equiv 3$$

$$\therefore y(1506) = \underline{\underline{3}}$$

5) B

Let $x \, ft^3$ = CO at time t minutes in the room

$$\frac{dx}{dt} = 0.15 \times 0.06 - \frac{0.15x}{1800} = 0.009 - \frac{1}{12000}x$$

$$\frac{dx}{dt} + \frac{1}{12000}x = 0.009$$

$$R = e^{\int \frac{1}{12000} dt} = e^{\frac{t}{12000}}$$

$$x = e^{\frac{-t}{12000}} \int 0.009 e^{\frac{t}{12000}} dt$$

$$= e^{\frac{-t}{12000}} \left\{ 108 e^{\frac{t}{12000}} + C \right\}$$

$$= 108 + Ce^{\frac{-t}{12000}}$$

$$x(0) = 0 \Rightarrow C = -108$$

$$\therefore x = 108 - 108e^{\frac{-t}{12000}}$$

$$x(a) = \frac{0.018}{100} \times 1800 = 0.324$$

$$\Rightarrow 0.324 = 108 - 108e^{\frac{-a}{12000}}$$

$$0.003 = 1 - e^{\frac{-a}{12000}}$$

$$e^{-\frac{a}{12000}} = 0.997$$

$$a = -12000 \ln 0.997$$

$$= 36.05\ldots$$

$$\approx \underline{\underline{36}}$$

6). B.

Let E = number of eggs at time t days

M = number of mayflies at time t days

$$\begin{cases} \dfrac{dE}{dt} = -k_E E \\ \dfrac{dM}{dt} = k_E E - k_M M \end{cases}$$

$$E = 1000000\, e^{-k_E t}$$

$$\frac{dM}{dt} + k_M M = 1000000\, k_E\, e^{-k_E t}$$

$$R = e^{\int k_M\, dt} = e^{k_M t}$$

$$M = e^{-k_M t} \int 1000000\, k_E\, e^{(k_M - k_E)t}\, dt$$

$$= \frac{1000000\, k_E}{k_M - k_E} e^{-k_E t} + C e^{-k_M t}$$

$$M(0) = 0 \Rightarrow C = -\frac{1000000\, k_E}{k_M - k_E}$$

$$\therefore M = \frac{1000000\, k_E}{k_M - k_E} \left\{ e^{-k_E t} - e^{-k_M t} \right\}$$

$$k_E = \frac{\ln 2}{2}, \quad k_M = \frac{\ln 2}{1}, \quad t = 3$$

$$\Rightarrow M = 1000000 \left\{ e^{-\frac{3}{2}\ln 2} - e^{-3\ln 2} \right\}$$

$$= 10^6 \left\{ 2^{-\frac{3}{2}} - 2^{-3} \right\}$$

7) C

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$\lambda = 1$ double root

$$\underline{\underline{y = C_1 e^x + C_2 x e^x}}$$

8) A

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, \text{ or } \lambda = 2$$

Try $y = (Ax^2 + Bx)e^{2x} = ue^{2x}$

$$y' = u'e^{2x} + 2ue^{2x}$$

$$y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}$$

$$u'' + u' = 2x - 1$$

$$2A + (2Ax + B) = 2x - 1$$

$$A = 1, \ B = -3$$

$$y = C_1 e^x + C_2 e^{2x} + (x^2 - 3x)e^{2x}$$

$$y' = C_1 e^x + 2C_2 e^{2x} + (2x - 3)e^{2x} + 2(x^2 - 3x)e^{2x}$$

$$\left.\begin{array}{l} y(0) = 3 \Rightarrow C_1 + C_2 = 3 \\ y'(0) = 1 \Rightarrow C_1 + 2C_2 = 4 \end{array}\right\} \Rightarrow C_1 = 2, \ C_2 = 1$$

$$\therefore y = 2e^x + e^{2x} + (x^2 - 3x)e^{2x}$$

$$y(1) = 2e + e^2 + (1 - 3)e^2 = 2e - e^2$$

$$= \underline{\underline{e(2 - e)}}$$

9) A

$$\lambda^2+1=0 \Rightarrow \lambda = \pm i$$

$$y''+y = 2\sin x \quad \text{---} \ ①$$

Let $v''+v = 2\cos x \quad \text{---} \ ②$

$$② + i① \Rightarrow z''+z = 2e^{ix}$$

where $z = v + iy$

Try $z = Axe^{ix}$

$$z' = Ae^{ix} + iAxe^{ix}$$

$$z'' = 2iAe^{ix} - Axe^{ix}$$

$$2iA = 2$$

$$A = -i$$

$$z = -ix(\cos x + i\sin x)$$

$$= x\sin x - ix\cos x$$

$$y = Im(z) = -x\cos x$$

$$\therefore y = C_1\cos x + C_2\sin x - x\cos x$$

$$y' = -C_1\sin x + C_2\cos x - \cos x + x\sin x$$

$$y(0)=0 \Rightarrow C_1 = 0$$

$$y'(0)=0 \Rightarrow C_2 = 1$$

$$\therefore y = \sin x - x\cos x$$

$$y\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{3\sqrt{3}-\pi}{6}$$

10) D

$$y_1 = \cos 3x, \quad y_2 = \sin 3x$$

$$v = \int \frac{r y_1}{y_1 y_2' - y_2 y_1'} dx$$

$$= \frac{1}{3} \int \frac{\cos 3x}{\sin 3x} dx$$

$$= \frac{1}{9} \ln|\sin 3x| + C$$

$$= \frac{1}{9} \ln(\sin 3x) + C \quad (\because 0 < x < \frac{\pi}{6})$$