2010/2011 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

March 2, 2011

8:30pm - 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Thirteen (13)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae Sheet

1. Integrating factor for $y'+Py=Q$ is given by

$$R=\exp\left(\int Pdx\right).$$

2. The variation of parameters formulae for $y''+py'+qy=r$:

$$u=\int\frac{-ry_2}{y_1y_2'-y_2y_1'}dx$$

$$v=\int\frac{ry_1}{y_1y_2'-y_2y_1'}dx\,.$$

1. Let y be a solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{x\,(x-1)}$$

such that

$$y(2) = -\ln 2.$$

Then $y\,(3) =$

(A) $\ln \frac{2}{3}$

(B) $-\ln \frac{2}{3}$

(**C**) $\ln \frac{4}{3}$

(**D**) 1

(**E**) None of the above

2. Let $y > 0$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

such that

$$y(1) = 1.$$

If $y\left(2\right) = a$, then a satisfies the equation

(A) $a^2 \ln a^2 = 4 + a^2$

(B) $a^2 \left(1 + \ln a^2\right) = 4$

(C) $a^2 \ln a^2 = a^2 \ln 2 + 4 + a^2$

(D) $a^2 \left(1 + \ln a^2\right) = a^2 \ln 4 + 4$

(E) None of the above

3. Let y be a solution of the differential equation

$$\frac{dy}{dt}\cos t + y\sin t = \tan t$$

such that

$$y(0) = 1.$$

Then $y\left(\frac{\pi}{4}\right) =$

(A) $\frac{1}{\sqrt{2}}$

(B) $\sqrt{2}$

(C) $\frac{3}{\sqrt{2}}$

(D) $\frac{3\sqrt{2}}{4}$

(E) None of the above

4. A roast beef, initially at 50° F, is placed in a 375° F oven at 5:00pm. At 6:15pm it is found that the temperature of the roast beef is 125° F. What time (correct to the nearest minute) should you remove the roast beef if you want it to be medium rare (i.e. its temperature is 150° F)?

(A) 6:45pm

(B) 6:40pm

(C) 6:38pm

(D) 6:33pm

(E) None of the above

5. A fossilized bone is found to contain 40% of the original amount of Carbon-14. We know that the half-life of Carbon-14 is 5600 years. Then the estimated age of the fossil to the nearest 100 years is equal to

(A) 6700 years

(B) 7100 years

(C) 7400 years

(D) 8100 years

(E) None of the above

6. The Jurong Lake has a volume of 700000 m^3. At time $t = 0$, the government starts a water cleaning process so that only fresh clean water flows into the lake. After 5 years, it is found that the pollution in the lake is reduced by 50%. If fresh water flows into the lake at a rate of r cubic metres per year and lake water flows out to the sea at the same rate, what is the value of r correct to the nearest thousands?

(A) 75000

(B) 83000

(C) 89000

(D) 97000

(E) None of the above

7. The general solution of the differential equation

$$y'' - 3\sqrt{2}y' + 4y = 0$$

is

(A) $y = c_1 e^{\sqrt{2}t} + c_2 e^{-2\sqrt{2}t}$

(B) $y = c_1 e^{\sqrt{2}t} + c_2 e^{2\sqrt{2}t}$

(C) $y = c_1 e^{2\sqrt{2}t} + c_2 t e^{\sqrt{2}t}$

(D) $y = c_1 e^{-\sqrt{2}t} + c_2 t e^{\sqrt{2}t}$

(E) None of the above

8. Let y be a solution of the differential equation

$$y'' - \frac{1}{x}y' = 0$$

such that

$$y(1) = 5,\ y'(1) = -1.$$

Find the value of $y(2)$.

(Hint: Use the substitution $w = y'$.)

(A) 2

(B) 2.5

(C) 3

(D) 3.5

(E) None of the above

9. Let y be a solution of the differential equation

$$y'' + 2y' - 3y = e^x$$

such that

$$y(0) = 0, \text{ and } y'(0) = 1.$$

Find the exact value of $y(2)$.

(A) $\frac{11e^8-3}{16e^6}$

(B) $\frac{3e^8-3}{16e^6}$

(C) $\frac{3e^8+3}{16e^6}$

(D) $\frac{11e^8+3}{16e^6}$

(E) None of the above

10. Let y be a solution of the differential equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{x},$$

such that

$$y(1) = 5e^3, \;\; y'(1) = 19e^3.$$

Then $y\,(2)$=

(A) $2e^3\,(4 + \ln 2)$

(B) $e^3\,(2 + \ln 4)$

(C) $e^6\,(8 + \ln 4)$

(D) $e^6\,(6 + \ln 2)$

(E) None of the above

END OF PAPER

Blank page for you to do your calculations

Answers to mid term test

1. A
2. B
3. D
4. A
5. C
6. D
7. B
8. D
9. A
10. C

MA1506 2011 Test Solutions

1). A

$$dy = \frac{dx}{x(x-1)} = \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$$

$$y = \ln|x-1| - \ln|x| + C = \ln\left|\frac{x-1}{x}\right| + C$$

$$y(2) = -\ln 2 \Rightarrow -\ln 2 = \ln\left|\frac{1}{2}\right| + C \Rightarrow C = 0$$

$$y(3) = \ln\frac{2}{3}$$

2). B

$$\text{Let } y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$x\frac{dv}{dx} + v = \frac{v}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v = -\frac{v^3}{1+v^2}$$

$$\left(\frac{1}{v^3} + \frac{1}{v}\right) dv = -\frac{dx}{x}$$

$$-\frac{1}{2v^2} + \ln|v| = -\ln|x| + C$$

$$-\frac{x^2}{2y^2} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C$$

$$y(1) = 1 \Rightarrow -\frac{1}{2} = C$$

$$-x^2 + 2y^2\ln|y| = -y^2$$

$$y(2) = a \Rightarrow -4 + 2a^2\ln|a| = -a^2$$

$$-4 + a^2\ln a^2 = -a^2$$

$$a^2(1 + \ln a^2) = 4$$

3). D

$$\frac{dy}{dt} + y\tan t = \sec t \tan t$$

$$R = e^{\int \tan t\, dt} = e^{-\ln \cos t} = \sec t$$

$$y = \cos t \int \sec^2 t \tan t\, dt$$

$$= \cos t \left\{ \frac{1}{2}\tan^2 t + C \right\}$$

$$y(0) = 1 \Rightarrow C = 1$$

$$y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left\{\frac{1}{2} + 1\right\} = \frac{3}{2\sqrt{2}} = \underline{\underline{\frac{3\sqrt{2}}{4}}}$$

4). A

$$\frac{dT}{dt} = k(T - 375) \Rightarrow T - 375 = Ae^{kt}$$

$$T(0) = 50 \Rightarrow T = 375 - 325e^{kt}$$

$$T(75) = 125 \Rightarrow -250 = -325e^{75k} \Rightarrow k = \frac{\ln 250 - \ln 325}{75}$$

$$T(t) = 150 \Rightarrow -225 = -325e^{kt}$$

$$\Rightarrow t = \frac{\ln 225 - \ln 325}{k}$$

$$= \frac{75(\ln 225 - \ln 325)}{\ln 250 - \ln 325} \approx 105.12$$

5 pm + 105 min. = $\underline{\underline{\text{6:45 pm}}}$

5). C

$$\frac{dQ}{dt} = kQ \Rightarrow Q = Q_0 e^{kt}$$

$$\frac{1}{2}Q_0 = Q_0 e^{5600k} \Rightarrow k = \frac{-\ln 2}{5600}$$

$$(0.4)Q_0 = Q_0 e^{kt} \Rightarrow t = \frac{\ln 0.4}{k} = \frac{5600 \ln 0.4}{-\ln 2}$$

$$\approx 7402.8$$

$$\approx \underline{\underline{7400}}$$

6). D

$$\frac{dQ}{dt} = -\frac{rQ}{700000} \Rightarrow Q = Q_0 e^{-\frac{rt}{700000}}$$

$$\frac{1}{2}Q_0 = Q_0 e^{-\frac{5r}{700000}}$$

$$\Rightarrow r = \frac{700000 \ln 2}{5}$$

$$\approx 97040.6$$

$$\approx \underline{\underline{97000}}$$

7). B

$$\lambda^2 - 3\sqrt{2}\lambda + 4 = 0$$

$$(\lambda - 2\sqrt{2})(\lambda - \sqrt{2}) = 0$$

$$y = C_1 e^{\sqrt{2}t} + C_2 e^{2\sqrt{2}t}$$

8). D

$$w' - \frac{1}{x} w = 0$$

$$\frac{dw}{w} = \frac{dx}{x} \Rightarrow \ln|w| = \ln|x| + C$$

$$\Rightarrow w = Ax$$

$$y'(1) = -1 \Rightarrow -1 = A$$

$$\therefore w = -x$$

$$dy = -x\,dx$$

$$y = -x^2/2 + B$$

$$y(1) = 5 \Rightarrow B = 5.5$$

$$\therefore y(2) = -2 + B = \underline{3.5}$$

9). A

$$\lambda^2+2\lambda-3=0$$

$$(\lambda+3)(\lambda-1)=0$$

Try $y = Axe^x$

$$y' = Ae^x + Axe^x$$

$$y'' = 2Ae^x + Axe^x$$

$$2Ae^x + 2Ae^x = e^x$$

$$A = \frac{1}{4}$$

$$y = C_1 e^x + C_2 e^{-3x} + \frac{1}{4}xe^x$$

$$y' = C_1 e^x - 3C_2 e^{-3x} + \frac{1}{4}e^x + \frac{1}{4}xe^x$$

$$\left.\begin{array}{l} y(0)=0 \Rightarrow C_1 + C_2 = 0 \\ y'(0)=1 \Rightarrow C_1 - 3C_2 + \frac{1}{4} = 1 \end{array}\right\} \Rightarrow C_1 = \frac{3}{16},\ C_2 = -\frac{3}{16}$$

$$y(2) = \frac{3}{16}e^2 - \frac{3}{16}e^{-6} + \frac{1}{2}e^2$$

$$= \frac{11e^8 - 3}{16e^6}$$

10). C

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0$$

$$y_1 = e^{3x}, \quad y_2 = xe^{3x}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

Try $y = uy_1 + vy_2$

$$u = \int \frac{-\frac{e^{3x}}{x} xe^{3x}}{e^{6x}} dx = -x$$

$$v = \int \frac{\frac{e^{3x}}{x} e^{3x}}{e^{6x}} dx = \ln|x|$$

$$y = -xe^{3x} + xe^{3x} \ln|x| + c_1 e^{3x} + c_2 xe^{3x}$$

$$y' = -e^{3x} - 3xe^{3x} + (\cdots)\ln|x| + e^{3x} + 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 xe^{3x}$$

$$y(1) = 5e^3 \Rightarrow 5e^3 = -e^3 + c_1 e^3 + c_2 e^3$$

$$y'(1) = 18e^3 \Rightarrow 18e^3 = -e^3 - 3e^3 + e^3 + 3c_1 e^3 + c_2 e^3 + 3c_2 e^3$$

$$\therefore c_1 + c_2 = 6$$

$$3c_1 + 4c_2 = 22$$

$$c_1 = \frac{\begin{vmatrix} 6 & 1 \\ 22 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}} = \frac{24 - 22}{4 - 3} = 2$$

$$c_2 = 4$$

$$y = 2e^{3x} + 4xe^{3x} - xe^{3x} + xe^{3x} \ln|x|$$

$$y(2) = 2e^{6} + 8e^{6} - 2e^{6} + 2e^{6} \ln 2$$

$$= 8e^{6} + e^{6} \ln 4$$

$$= \underline{\underline{e^{6}(8 + \ln 4)}}$$