2008/2009 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

March 4, 2009

8:30pm - 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Fifteen (15)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).
4. Use **only 2B pencils** for FORM CC1.
5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles carefully. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. Write your full name in section A of FORM CC1.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be properly and completely shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1.
11. Submit FORM CC1 before you leave the test hall.

Formulae Sheet

1. Integrating factor for $y' + Py = Q$ is given by

$$R = \exp\left(\int P dx\right).$$

2. The variation of parameters formulae for $y'' + py' + qy = r$:

$$u = \int \frac{-ry_2}{y_1 y_2' - y_2 y_1'} dx$$

$$v = \int \frac{ry_1}{y_1 y_2' - y_2 y_1'} dx.$$

1. Let y be a solution of the differential equation

$$(\sec x)\,\frac{dy}{dx} = y$$

such that

$$y(0) = 1.$$

Then $y\left(\frac{\pi}{6}\right) =$

(A) $\sqrt{e}$

(B) $\frac{1}{2}e$

(**C**) 0

(**D**) e

(**E**) $\frac{1}{2}\sqrt{e}$

2. Let $y > 0$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

such that

$$y(1) = 1.$$

Then $y\,(2) =$

(A) $\sqrt{10} + 2$

(B) $\sqrt{5} - 1$

(**C**) 2

(**D**) $\sqrt{5} + 1$

(**E**) $\sqrt{10} - 2$

3. Let y be a solution of the differential equation

$$(1+e^x)\,y' + 2e^x y = (1+e^x)\,e^x$$

such that

$$y(1) = 1.$$

Then $y\,(0) =$

(A) $\frac{9+e+e^3}{15}$

(B) $\frac{10-e-e^3}{12}$

(C) $\frac{9+3e-e^3}{4}$

(D) $\frac{10+3e-e^3}{12}$

(E) $\frac{10+e+e^3}{12}$

4. In a piece of wood, it was found that 85% of the carbon-14 had decayed. Using the fact that the half life of carbon-14 is 5730 years, determine the age of the wood.

(A) 12435

(B) 13870

(C) 14741

(D) 15683

(E) 16579

5. A copper ball with an initial temperature of 100 degrees is plunged into a bath of water which is maintained at a temperature of 30 degrees. The rate of change of temperature is proportional to the difference between the temperature of the ball and that of the water, with a constant of proportionality $-k$ (in units of 1/minute). After one minute, the temperature of the ball is 50 degrees. Then $k =$

(A) e^{-1}

(B) $\ln\left(\frac{5}{3}\right)$

(C) $\ln\left(\frac{7}{2}\right)$

(D) $\ln\left(\frac{7}{3}\right)$

(E) $\ln\left(\frac{7}{5}\right)$

6. An object of mass 1 kg, acted upon by gravity, falls from rest at time $t = 0$ in a medium having a resistance force $(9.8)\,v^2$ newtons, where the velocity v is measured in meters per second. Taking the gravitational constant $g = 9.8$ meters per second squared, find its velocity at a time t seconds later.

(A) $(4.9)\,e^{(9.8)t}$

(B) $\tanh(9.8t)$

(C) $\sinh(9.8t)$

(D) $\cosh(9.8t)$

(E) $e^{(9.8)t} - e^{-(9.8)t}$

7. Which one of the following equations has $y_1 = 3e^{2x}$ and $y_2 = -4e^{-x}$ as two solutions?

(A) $y'' - y' - 2y = 0$

(B) $y'' + y' - 2y = 0$

(C) $y'' - 4y' + 4y = 0$

(D) $y'' + 3y' - 4y = 0$

(E) $y'' + y' - 12y = 0$

8. Let y be a solution of the differential equation

$$y'' - y = -x$$

such that

$$y(0) = y'(0) = 0.$$

Then $\lim_{x \to 1} y(x) =$

(A) $\frac{e}{2} - \frac{e^{-1}}{2} + 1$

(B) $\frac{e^{-1}}{2} - \frac{e}{2} + 1$

(C) $\frac{e^{-1}}{2} - \frac{e}{2} - 1$

(D) $\frac{e}{2} - \frac{e^{-1}}{2} - 1$

(E) 1

9. Let y be a solution of the differential equation

$$y'' - 4y' + 3y = (x+1)e^{3x}$$

such that

$$y(0) = 2, \quad y'(0) = \frac{17}{4}.$$

Then $y(4) =$

(A) $5e^{12} + e^4$

(B) $5e^{12} - e^4$

(C) $2e^{12} + 6e^4$

(D) $8e^{12} - e^4$

(E) $6e^{12} + e^4$

10. Let y be a solution of the differential equation

$$y'' + 4y = \tan 2x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4},$$

such that

$$y(0) = -1, \;\; y'(0) = \frac{3}{2}.$$

Then $y\left(\frac{\pi}{6}\right) =$

(A) $\frac{1}{4}\left[\sqrt{3} + 1 - \ln\left(2 + \sqrt{3}\right)\right]$

(B) $\frac{1}{2}\left[\sqrt{3} - 1 + \ln\left(2 - \sqrt{3}\right)\right]$

(C) $\frac{1}{8}\left[4(\sqrt{3} - 1) - \ln\left(2 + \sqrt{3}\right)\right]$

(D) $\frac{1}{8}\left[4(\sqrt{3} - 1) - \ln\left(2 - \sqrt{3}\right)\right]$

(E) $\frac{1}{8}\left[4(\sqrt{3} + 1) + \ln\left(2 - \sqrt{3}\right)\right]$

END OF PAPER

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Blank page for you to do your calculations

Blank page for you to do your calculations

Answers to mid term test

1. A
2. E
3. D
4. D
5. C
6. B
7. A
8. B
9. E
10. C

1). A

$$\frac{dy}{y} = \frac{dx}{\sec x} = \cos x\,dx \Rightarrow \ln|y| = \sin x + C$$

$$\Rightarrow y = Ae^{\sin x}$$

$$y(0)=1 \Rightarrow 1 = A \Rightarrow y = e^{\sin x}$$

$$y(\tfrac{\pi}{6}) = e^{\frac{1}{2}} = \underline{\underline{\sqrt{e}}}$$

2). E

Let $y = ux \Rightarrow y' = u'x + u$

$$\therefore \quad u'x + u = \frac{x - ux}{x + ux} = \frac{1-u}{1+u}$$

$$u'x = \frac{1-u}{1+u} - u = \frac{1-u-u-u^2}{1+u} = \frac{1-2u-u^2}{1+u}$$

$$\Rightarrow \frac{1+u}{1-2u-u^2}\,du = \frac{dx}{x} \Rightarrow \frac{(u+1)\,du}{u^2+2u-1} = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln|u^2+2u-1| = -\ln|x| + C_1 = \ln\frac{C}{|x|}$$

$$\therefore \quad u^2+2u-1 = \frac{A}{x^2}$$

$$\therefore \quad y^2+2xy-x^2 = A$$

$$y(1)=1 \Rightarrow 1+2-1 = A \Rightarrow A = 2$$

$$\therefore \quad y^2+2xy-x^2 = 2$$

$$x=2 \Rightarrow y^2+4y-6 = 0$$

$$y = \frac{-4 \pm \sqrt{16+24}}{2} = -2 \pm \sqrt{10}$$

$$\because y > 0 \quad \therefore \quad \underline{\underline{y = \sqrt{10} - 2}}$$

3). D

$$y' + \frac{2e^x}{1+e^x}y = e^x$$

$$R = e^{\int \frac{2e^x}{1+e^x}dx} = e^{2\ln(1+e^x)} = (1+e^x)^2$$

$$y = \frac{1}{(1+e^x)^2}\int (1+e^x)^2 e^x dx = \frac{1}{(1+e^x)^2}\left\{\frac{1}{3}(1+e^x)^3 + c\right\}$$

$$y(1)=1 \Rightarrow 1 = \frac{1}{(1+e)^2}\left\{\frac{1}{3}(1+e)^3 + c\right\}$$

$$\Rightarrow c = (1+e)^2 - \frac{1}{3}(1+e)^3$$

$$\therefore y(0) = \frac{1}{2^2}\left\{\frac{1}{3}(2)^3 + c\right\} = \frac{1}{12}\left\{8 + 3(1+e)^2 - (1+e)^3\right\}$$

$$= \frac{1}{12}(8 + 3 + 6e + 3e^2 - 1 - 3e - 3e^2 - e^3)$$

$$= \underline{\underline{\frac{10+3e-e^3}{12}}}$$

4). D

$$\frac{dQ}{dt} = kQ \Rightarrow Q = Ae^{kt}$$

$$\frac{1}{2}A = Ae^{k(5730)} \Rightarrow k = \frac{-\ln 2}{5730}$$

$$0.15A = Ae^{kT} \Rightarrow T = \frac{\ln 0.15}{k} = -\frac{5730\ln 0.15}{\ln 2}$$

$$= \underline{\underline{15683}}$$

5). C

$$\frac{dT}{dt} = -k(T-30) \Rightarrow T-30 = Ae^{-kt}$$

$$T(0)=100 \Rightarrow 100-30 = A \Rightarrow A = 70$$

$$\therefore T = 30 + 70e^{-kt}$$

$$50 = 30 + 70e^{-k} \Rightarrow 20 = 70e^{-k}$$

$$\Rightarrow k = \ln\frac{7}{2}$$

6). B

$$m\frac{dv}{dt} = mg - 9.8v^2 = 9.8(m - v^2)$$

$$m = 1\,kg \Rightarrow \frac{dv}{dt} = 9.8(1-v^2)$$

$$\Rightarrow \frac{dv}{(1-v)(1+v)} = 9.8\,dt$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{1-v} + \frac{1}{1+v}\right)dv = 9.8\,dt$$

$$\therefore \frac{1}{2}\left\{-\ln|1-v| + \ln|1+v|\right\} = 9.8t + C$$

$$\frac{1+v}{1-v} = Ae^{19.6t}$$

$$v(0)=0 \Rightarrow A=1 \Rightarrow \frac{1+v}{1-v} = e^{19.6t}$$

$$1+v = e^{19.6t} - ve^{19.6t}$$

$$v = \frac{e^{19.6t}-1}{e^{19.6t}+1} = \frac{e^{9.8t} - e^{-9.8t}}{e^{9.8t} + e^{-9.8t}}$$

$$= \tanh 9.8t$$

7). A

Characteristic equation is $(\lambda-2)(\lambda+1)=0$

i.e. $\lambda^2-\lambda-2=0$

$\therefore\ y''-y'-2y=0$

8). B

$\lambda^2-1=0 \Rightarrow \lambda=\pm 1$

Try $y=Ax+B \Rightarrow -Ax-B=-x \Rightarrow A=1,\ B=0$

$\therefore\ y=C_1e^{x}+C_2e^{-x}+x \Rightarrow 0=C_1+C_2$

$y'=C_1e^{x}-C_2e^{-x}+1 \Rightarrow 0=C_1-C_2+1$

$\therefore\ C_1=-\frac{1}{2},\ C_2=\frac{1}{2}$

$\therefore\ y=-\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}+x \Rightarrow \lim_{x\to 1} y(x)=-\frac{1}{2}e+\frac{1}{2}e^{-1}+1$

9). E

$\lambda^2-4\lambda+3=0 \Rightarrow \lambda=1,3.$

Try $y=(Ax^2+Bx+C)e^{3x}$ $\therefore\ y'=(2Ax+B)e^{3x}+3(Ax^2+Bx+C)e^{3x}$

$y''=2Ae^{3x}+6(2Ax+B)e^{3x}+9(Ax^2+Bx+C)e^{3x}$

$\therefore\ 2A+2(2Ax+B)=x+1 \Rightarrow A=\frac{1}{4},\ B=\frac{1}{4}$

$\therefore\ y=C_1e^{x}+C_2e^{3x}+\left(\frac{1}{4}x^2+\frac{1}{4}x\right)e^{3x} \Rightarrow 2=C_1+C_2$

$y'=C_1e^{x}+3C_2e^{3x}+\left(\frac{1}{2}x+\frac{1}{4}\right)e^{3x}+\left(\frac{1}{4}x^2+\frac{1}{4}x\right)3e^{3x} \Rightarrow \frac{17}{4}=C_1+3C_2+\frac{1}{4}$

$\therefore\ C_1=C_2=1$

$y=e^{x}+e^{3x}+\left(\frac{1}{4}x^2+\frac{1}{4}x\right)e^{3x}$

$y(4)=e^4+e^{12}+(4+1)e^{12}=e^4+6e^{12}$

10). C

$$y'' + 4y = \tan 2x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

Let $y_1 = \cos 2x$, $y_2 = \sin 2x$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u = \int \frac{-\sin 2x \tan 2x \, dx}{2} = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx = \frac{1}{4}\sin 2x - \frac{1}{4}\ln|\sec 2x + \tan 2x|$$

$$v = \int \frac{\cos 2x \tan 2x}{2} dx = -\frac{1}{4}\cos 2x$$

$$\therefore y = C_1 \cos 2x + C_2 \sin 2x + \left(\frac{1}{4}\sin 2x - \frac{1}{4}\ln|\sec 2x + \tan 2x|\right)\cos 2x$$
$$- \frac{1}{4}\cos 2x \sin 2x$$

$$\therefore y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \ln(\sec 2x + \tan 2x)$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x + \frac{1}{2}\sin 2x \ln(\sec 2x + \tan 2x)$$
$$- \frac{\cos 2x}{4} \frac{1}{\sec 2x + \tan 2x}(2\sec 2x \tan 2x + 2\sec^2 2x)$$

$$y(0) = -1 \Rightarrow -1 = C_1$$

$$y'(0) = \frac{3}{2} \Rightarrow \frac{3}{2} = 2C_2 - \frac{1}{4}\left(\frac{1}{1}\right)(2) \Rightarrow C_2 = 1$$

$$\therefore y = -\cos 2x + \sin 2x - \frac{1}{4}\cos 2x \ln(\sec 2x + \tan 2x)$$

$$y\left(\frac{\pi}{6}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} - \frac{1}{4}\left(\frac{1}{2}\right)\ln(2+\sqrt{3})$$

$$= \frac{1}{8}\left\{4(\sqrt{3}-1) - \ln(2+\sqrt{3})\right\}$$