2007/2008 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

March 4, 2008

8:00pm - 9:00pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Twelve (12)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).
4. Use **only 2B pencils** for FORM CC1.
5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles carefully. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. Write your full name in section A of FORM CC1.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be properly and completely shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1.
11. Submit FORM CC1 before you leave the test hall.

Formulae Sheet

1. Integrating factor for $y' + Py = Q$ is given by

$$R = \exp\left(\int P dx\right).$$

2. The variation of parameters formulae for $y'' + py' + qy = r$:

$$u = \int \frac{-ry_2}{y_1 y_2' - y_2 y_1'} dx$$

$$v = \int \frac{ry_1}{y_1 y_2' - y_2 y_1'} dx .$$

1. Let y be a solution of the differential equation

$$\frac{dy}{dx} = x^3 y^2$$

such that

$$y(0) = 1.$$

Then $y(1) =$

(A) 3

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) $\frac{4}{3}$

(E) 4

2. A certain substance grows at a rate proportional to the amount present. Initially there are 250 gm of this substance and it grows to 800 gm after 7 hours. How long from the start does it take for the substance to grow to 1600 gm?

(A) 10.8 hours

(B) 12.6 hours

(C) 14 hours

(D) 10.5 hours

(E) 11.2 hours

3. Let y be a solution of the differential equation

$$xy' + (x+2)\,y = 3xe^{-x}, x > 0,$$

such that

$$y(1) = 0.$$

Then $y\,(\ln 2) =$

(A) $\frac{(\ln 2)^3-1}{2(\ln 2)^2}$

(B) $\frac{(\ln 2)^2+1}{2(\ln 2)^3}$

(C) $\frac{1-(\ln 2)^3}{2(\ln 2)^2}$

(D) $\frac{2(\ln 2)^2+1}{(\ln 2)^3}$

(E) $\frac{2(\ln 2)^2-1}{(\ln 2)^3}$

4. An object of unknown temperature is placed, at time $t = 0$ minutes, in a refrigerator which maintains a constant temperature of 0^o F. At $t = 20$ minutes, the temperature of the object is 40^o F. At $t = 40$ minutes, the temperature of the object is 20^o F. What is the initial temperature of this object?

(A) 100^o F

(B) 80^o F

(C) 120^o F

(D) 60^o F

(E) 90^o F

5. Let y be a solution of the differential equation

$$\frac{dy}{dx} + y = y^3$$

such that

$$y(0) = 1.$$

Then $y\,(50) =$

(A) 1

(B) $1 + e^{50}$

(C) $1 + e^{-50}$

(D) $1 - e^{-50}$

(E) 51

6. You started an experiment with a tank containing 1000 gal of water in which 100 lb of salt is dissolved. At time $t = 0$, you turned on a tap to let in a salt solution at the rate of 10 gal per minute, and each gallon contained 2 lb of dissolved salt. The mixture in the tank was kept uniform by stirring and this well-stirred solution was let out at the same rate of 10 gal per minute when the tap was turned on. Find the amount of salt in the tank at time $t = 2$ hours.

(A) 1367.5 lb

(B) 1743.9 lb

(C) 1427.7 lb

(D) 137.62 lb

(E) 314.85 lb

7. Let y be a solution of the differential equation

$$2y'' + y' - y = e^{\frac{1}{2}x}$$

such that

$$y(0) = 3, \quad y'(0) = \frac{1}{3}.$$

Then $y(2) =$

(A) $\frac{4}{3}e + \frac{1}{e}$

(B) $2e + \frac{1}{e^2}$

(C) $\frac{8}{5}e + \frac{1}{e}$

(D) $\frac{8}{3}e + \frac{1}{e^2}$

(E) $\frac{8}{5}e + \frac{1}{e^2}$

8. Let y be a solution of the differential equation

$$4y'' + 4y' + 5y = 0$$

such that

$$y(0) = 3, \quad y'(0) = \frac{1}{2}.$$

Then $y(\frac{\pi}{4}) =$

(A) $\frac{5\sqrt{2}}{2}e^{-\frac{\pi}{4}}$

(B) $\frac{5\sqrt{2}}{2}e^{-\frac{\pi}{8}}$

(C) $\frac{-3\sqrt{2}}{2}e^{-\frac{\pi}{4}}$

(D) $\frac{3\sqrt{2}}{2}e^{-\frac{\pi}{4}}$

(E) $\frac{3\sqrt{2}}{2}e^{-\frac{\pi}{8}}$

9. Let y be a solution of the differential equation

$$y'' + y' - 2y = -6x^2 + 14x - 8$$

such that

$$y(0) = 1, \quad y'(0) = 1.$$

Then $y(2) =$

(A) $e^2 - 4e^{-4} + 5$

(B) $-3e^2 + 2e^{-4} + 7$

(C) $e^2 - 3e^{-4} + 11$

(D) $5e^2 - e^{-4} + 1$

(E) $-e^2 - 3e^{-4} + 9$

10. Let y be a solution of the differential equation

$$y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0,$$

such that

$$y(1) = y'(1) = 0.$$

Then $y\,(2)$=

(A) $[(\ln 4) - 1]\, e^2$

(B) $[(\ln 2) + 1]\, e^2$

(C) $[(2\ln 8) - 2]\, e^2$

(D) $[(2\ln 2) + 1]\, e^2$

(E) $[(2\ln 4) - 1]\, e^2$

END OF PAPER

Answers to mid term test

1. D
2. E
3. A
4. B
5. A
6. C
7. D
8. B
9. E
10. A

MA1506 Mid-term test solutions

1). D.

$$y^{-2}dy = x^3 dx \Rightarrow -y^{-1} = \frac{1}{4}x^4 + C \Rightarrow y = \frac{-1}{\frac{1}{4}x^4 + C}$$

$$y(0) = 1 \Rightarrow C = -1 \Rightarrow y = \frac{1}{1 - \frac{1}{4}x^4}$$

$$\therefore y(1) = \underline{\underline{\frac{4}{3}}}$$

2). E.

$$\frac{dx}{dt} = kx \Rightarrow \frac{dx}{x} = k\,dt \Rightarrow \ln|x| = kt + C \Rightarrow x = Ae^{kt}$$

$$x(0) = 250 \Rightarrow A = 250 \Rightarrow x = 250e^{kt}$$

$$x(7) = 800 \Rightarrow 800 = 250e^{7k} \Rightarrow k = \frac{1}{7}\ln\frac{800}{250}$$

$$1600 = 250e^{kt} \Rightarrow kt = \ln\frac{1600}{250} \Rightarrow t = \frac{\ln 1600 - \ln 250}{k}$$

$$\Rightarrow t = \frac{7(\ln 1600 - \ln 250)}{\ln 800 - \ln 250} \approx \underline{\underline{11.2}}$$

3). A.

$$y' + \frac{x+2}{x}y = 3e^{-x}$$

$$R = e^{\int \frac{x+2}{x}dx} = e^{\int (1 + \frac{2}{x})dx} = e^{x + 2\ln x} = e^x e^{\ln x^2} = x^2 e^x$$

$$y = \frac{1}{x^2 e^x}\int x^2 e^x (3e^{-x})\,dx = \frac{1}{x^2 e^x}\int 3x^2 dx = \frac{e^{-x}}{x^2}(x^3 + C)$$

$$y(1) = 0 \Rightarrow 0 = e^{-1}(1 + C) \Rightarrow C = -1$$

$$\therefore y = \frac{x}{e^x} - \frac{1}{x^2 e^x}$$

$$y(\ln 2) = \frac{\ln 2}{2} - \frac{1}{(\ln 2)^2 (2)} = \underline{\underline{\frac{(\ln 2)^3 - 1}{2(\ln 2)^2}}}$$

4). B. $\frac{dT}{dt} = k(T-0) = kT \Rightarrow T = Ae^{kt}$

$T(20) = 40 \Rightarrow 40 = Ae^{20k} \Rightarrow 20k = \ln 40 - \ln A$

$T(40) = 20 \Rightarrow 20 = Ae^{40k} \Rightarrow 40k = \ln 20 - \ln A$

$\Rightarrow 0 = 2\ln 40 - \ln 20 - \ln A$

$\Rightarrow \ln A = \ln \frac{40^2}{20} = \ln 80 \Rightarrow \underline{\underline{A = 80}}$

5). A. Let $z = y^{1-3} = y^{-2} \Rightarrow z' = -2y^{-3}y' \Rightarrow y' = -\frac{1}{2}y^3 z'$

$\therefore -\frac{1}{2}y^3 z' + y = y^3 \Rightarrow z' - 2y^{-2} = -2 \Rightarrow z' - 2z = -2$

$R = e^{\int -2dx} = e^{-2x}$

$z = \frac{1}{e^{-2x}} \int e^{-2x}(-2)dx = e^{2x}(e^{-2x} + c) = 1 + ce^{2x}$

$\therefore \frac{1}{y^2} = 1 + ce^{2x}$; $y(0) = 1 \Rightarrow 1 = 1 + C \Rightarrow C = 0$

$\therefore y^2 \equiv 1$

$\therefore y \equiv 1$ or $y \equiv -1$ ($\because y$ is differentiable, $\therefore$ continuous)

$\because y(0) = 1 \quad \therefore y \equiv 1$

$\therefore \underline{\underline{y(50) = 1}}$

6). C. $\frac{dx}{dt} = 20 - 10\frac{x}{1000} = 20 - 0.01x = 0.01(2000 - x)$

$\frac{dx}{2000 - x} = 0.01dt \Rightarrow -\ln|2000 - x| = 0.01t + C$

$2000 - x = Ae^{-0.01t}$

$x = 2000 - Ae^{-0.01t}$

$x(0) = 100 \Rightarrow A = 1900 \Rightarrow x = 2000 - 1900e^{-0.01t}$

2 hours = 120 min.

$x(120) = 2000 - 1900e^{-1.2} = \underline{\underline{1427.7}}$

7). D. $2y''+y'-y=0$

$2\lambda^2+\lambda-1=0$

$(2\lambda-1)(\lambda+1)=0 \Rightarrow \lambda=\frac{1}{2}$ or $\lambda=-1$.

Try $y=(Ax+B)e^{\frac{1}{2}x}$

$y'=Ae^{\frac{1}{2}x}+\frac{1}{2}(Ax+B)e^{\frac{1}{2}x}$

$y''=\frac{1}{2}Ae^{\frac{1}{2}x}+\frac{1}{2}Ae^{\frac{1}{2}x}+\frac{1}{4}(Ax+B)e^{\frac{1}{2}x}$

$=Ae^{\frac{1}{2}x}+\frac{1}{4}(Ax+B)e^{\frac{1}{2}x}$

$\therefore 2y''+y'-1=e^{\frac{1}{2}x}$

$\Rightarrow 3Ae^{\frac{1}{2}x}=e^{\frac{1}{2}x} \Rightarrow A=\frac{1}{3}$

$\therefore y=c_1e^{\frac{1}{2}x}+c_2e^{-x}+\frac{1}{3}xe^{\frac{1}{2}x}$

$y'=\frac{1}{2}c_1e^{\frac{1}{2}x}-c_2e^{-x}+\frac{1}{3}e^{\frac{1}{2}x}+\frac{1}{6}xe^{\frac{1}{2}x}$

$y(0)=3 \Rightarrow 3=c_1+c_2$

$y'(0)=\frac{1}{3} \Rightarrow \frac{1}{3}=\frac{1}{2}c_1-c_2+\frac{1}{3} \Rightarrow 0=\frac{1}{2}c_1-c_2$

$\Rightarrow c_1=2,\ c_2=1.$

$\therefore y=2e^{\frac{1}{2}x}+e^{-x}+\frac{1}{3}xe^{\frac{1}{2}x}$

$y(2)=2e+e^{-2}+\frac{2}{3}e$

$=\frac{8}{3}e+\frac{1}{e^2}$

8). B. $4y'' + 4y' + 5y = 0$

$$4\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 80}}{8} = \frac{-4 \pm 8i}{8} = -\frac{1}{2} \pm i$$

$$y = c_1 e^{-\frac{1}{2}x} \cos x + c_2 e^{-\frac{1}{2}x} \sin x$$

$$y' = -\frac{1}{2} c_1 e^{-\frac{1}{2}x} \cos x - c_1 e^{-\frac{1}{2}x} \sin x$$
$$-\frac{1}{2} c_2 e^{-\frac{1}{2}x} \sin x + c_2 e^{-\frac{1}{2}x} \cos x$$

$y(0) = 3 \Rightarrow 3 = c_1$

$y'(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = -\frac{1}{2} c_1 + c_2 \Rightarrow c_2 = 2$

$$\therefore y = 3e^{-\frac{1}{2}x} \cos x + 2e^{-\frac{1}{2}x} \sin x$$

$$y\left(\tfrac{\pi}{4}\right) = 3e^{-\frac{\pi}{8}}\left(\tfrac{1}{\sqrt{2}}\right) + 2e^{-\frac{\pi}{8}}\left(\tfrac{1}{\sqrt{2}}\right)$$

$$= \frac{5}{\sqrt{2}} e^{-\frac{\pi}{8}}$$

$$= \frac{5\sqrt{2}}{2} e^{-\frac{\pi}{8}}$$

9). E. $\lambda^2+\lambda-2=0$

$\Rightarrow (\lambda+2)(\lambda-1)=0 \Rightarrow \lambda=-2$ or $\lambda=1$

Try $y = Ax^2+Bx+C$

$y' = 2Ax+B$

$y'' = 2A$

$\therefore 2A+(2Ax+B)-2(Ax^2+Bx+C) = -6x^2+14x-8$

Compare $x^2 \Rightarrow -2A=-6 \Rightarrow A=3$

compare $x \Rightarrow 2A-2B=14 \Rightarrow B=-4$

compare constant $\Rightarrow 2A+B-2C=-8 \Rightarrow C=5$

$\therefore y = C_1e^{-2x}+C_2e^x+3x^2-4x+5$

$y' = -2C_1e^{-2x}+C_2e^x+6x-4$

$y(0)=1 \Rightarrow 1 = C_1+C_2+5$

$y'(0)=1 \Rightarrow 1=-2C_1+C_2+0-4$

$\therefore C_1=-3, \quad C_2=-1$

$y=-3e^{-2x}-e^x+3x^2-4x+5$

$y(2)=-3e^{-4}-e^2+12-8+5$

$= \underline{\underline{-e^2-3e^{-4}+9}}$

10). A. $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$ double root.

Let $y_1 = e^x$, $y_2 = xe^x$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

$$u = \int \frac{-\frac{e^x}{x} xe^x}{e^{2x}} dx = \int \frac{-e^{2x}}{e^{2x}} dx = -x$$

$$v = \int \frac{\frac{e^x}{x} e^x}{e^{2x}} dx = \int \frac{1}{x} dx = \ln|x|$$

$\therefore y = C_1 e^x + C_3 xe^x - xe^x + x(\ln|x|)e^x$

$\therefore y = C_1 e^x + C_2 xe^x + x(\ln|x|)e^x$

$\therefore y' = C_1 e^x + C_2 e^x + C_2 xe^x + (\ln|x|)e^x + x(\frac{1}{x})e^x + x(\ln|x|)e^x$

$y(1) = 0 \Rightarrow C_1 e + C_2 e = 0 \Rightarrow C_1 + C_2 = 0$

$y'(1) = 0 \Rightarrow C_1 e + 2C_2 e + e = 0 \Rightarrow C_1 + 2C_2 = -1$

$\therefore C_1 = 1, \quad C_2 = -1$

$\therefore y = e^x - xe^x + x(\ln|x|)e^x$

$y(2) = 2(\ln 2)e^2 - e^2 = \underline{\underline{[(\ln 4) - 1]e^2}}$

5). A (Second solution)

This can be solved more easily by using the materials we learned in Chapter 3.

$$\frac{dy}{dx} = y^3 - y = y(y^2-1) = y(y+1)(y-1).$$

$\therefore$ $y=-1$, $y=0$, $y=1$ are equilibrium solutions.

$\because$ y^3-y is continuously differentiable

$\therefore$ the NO CROSSING RULE is applicable.

$\because$ $y(0)=1$

$\therefore$ $y \equiv 1$

i.e. $y(50) = \underline{\underline{1}}$