2006/2007 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

February 26, 2007

SESSION 1 : 6:00 - 7:00pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Twelve (12)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).
4. Use **only 2B pencils** for FORM CC1.
5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles carefully. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. Write your full name in section A of FORM CC1.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be properly and completely shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1.
11. Submit FORM CC1 before you leave the test hall.

Formulae Sheet

1. Integrating factor for $y' + Py = Q$ is given by

$$R = \exp\left(\int P dx\right).$$

2. The variation of parameters formulae for $y'' + py' + qy = r$:

$$u = \int \frac{-ry_2}{y_1 y_2' - y_2 y_1'} dx$$

$$v = \int \frac{ry_1}{y_1 y_2' - y_2 y_1'} dx.$$

1. Let $f(x)$ be a solution of the differential equation $f'(x) = \frac{x+1}{\sqrt{x}}$, $x > 0$, such that $f(1) = 2$. Then $f(4) =$

(A) 8

(B) $\frac{22}{3}$

(C) $\frac{17}{2}$

(D) $\frac{26}{3}$

(E) $\frac{5}{2}$

2. Let y be a solution of the differential equation

$$\left(x^2+y^2\right)+\left(x^2-xy\right)y'=0,\ \ x>0,$$

such that

$$y(1)=0.$$

If $y\,(2)=a$, then a satisfies the equation

(A) $a^2-4a+4-2e^{\frac{a}{2}}=0$

(B) $a^2+4a+4-2e^{\frac{a}{2}}=0$

(C) $-a^2+4a+4-2e^{\frac{a}{2}}=0$

(D) $a^2+4a-4-2e^{\frac{a}{2}}=0$

(E) $a^2+4a-4+2e^{\frac{a}{2}}=0$

3. Let y be a solution of the differential equation $\frac{1}{x}y' - 2y = \frac{1}{x}e^{x^2}$, $x > 0$, such that $y(1) = 2e$. Then $y(2) =$

(A) $3e^4$

(B) $2e^4$

(C) $5e^4$

(D) e^4

(E) $\frac{5}{8}e^4$

4. Let y be a solution of the differential equation

$$2xyy' = y^2 - 2x^3, \ x > 0,$$

such that

$$y(1) = 2.$$

If $y\left(\frac{1}{2}\right) = a$, then a satisfies the equation

(A) $a^2 = \frac{17}{8}$

(B) $a^2 = \frac{17}{4}$

(C) $a^2 = \frac{9}{4}$

(D) $a^2 = \frac{19}{4}$

(E) $a^2 = \frac{19}{8}$

5. Experiments show that the rate of change of the temperature of a small iron ball is proportional to the difference between its temperature and the temperature of its environment. The ball is heated to x°F and then left to cool in a room which has a constant temperature of 70°F. Its temperature falls to 170°F after one hour. After 3 hours of cooling, its temperature is 80°F. Then x is approximately equal to

(A) 358

(B) 300

(C) 401

(D) 291

(E) 386

6. At time $t = 0$, you started an experiment with a certain amount of a substance which decays at a rate proportional to the amount present. At $t = 100$, you found that 10% of the substance was gone. At $t = T$ you found that half of the substance was gone. What is the approximate value of T?

(A) 917

(B) 409

(C) 573

(D) 311

(E) 658

7. The general solution of $y'' - 0.01y = 0$ is

(A) $y = Ae^{0.01x} + Be^{-0.01x}$

(B) $y = Ae^{0.1x} + Bxe^{-0.1x}$

(C) $y = Ae^{0.1x} + Be^{-0.1x}$

(D) $y = Axe^{0.1x} + Be^{-0.1x}$

(E) $y = A\cos(0.1x) + B\sin(0.1x)$

8. Let $y(t)$ be a solution of the differential equation

$$2y''(t) + 13y'(t) + 15y(t) = 0$$

such that

$$y(1) = 1, \quad y(0) = 0.$$

Then $\lim_{t \to +\infty} y(t) =$

(A) 2

(B) $e^{-2} - e^{-3}$

(C) 0

(D) ∞

(E) $-\infty$

9. Let $y(x)$ be a solution of the differential equation

$$y''(x) - 3y'(x) + 2y(x) = xe^x$$

such that

$$y(0) = 0, \quad y'(0) = -2.$$

Then $y(1) =$

(A) $e^2 - \frac{1}{2}e$

(B) $-e^2 + \frac{1}{2}e$

(C) $e^2 - e$

(D) $\frac{1}{2}e^2 - e$

(E) $-e^2 - \frac{1}{2}e$

10. Let $y(x)$ be a solution of the differential equation

$$y''(x) + 2y'(x) + y(x) = 2 + e^{2x}$$

such that

$$y(0) = \frac{37}{9}, \quad y'(0) = -\frac{7}{9}.$$

Then $y(-1)$=

(A) $e - 2 + \frac{1}{9e^2}$

(B) $\frac{1}{2}e + 2 + \frac{1}{e}$

(C) $e + 2 + \frac{1}{9e^2}$

(D) $e + 2 - \frac{1}{9e^2}$

(E) $e^2 - 2 + \frac{1}{2e}$

END OF PAPER

Answers to mid term test for session 1

1. D
2. B
3. A
4. E
5. E
6. E
7. C
8. C
9. E
10. C

Session 1

1). $f'(x) = \sqrt{x} + \frac{1}{\sqrt{x}} \Rightarrow f(x) = \frac{2}{3}x^{3/2} + 2x^{\frac{1}{2}} + C$

$f(1) = 2 \Rightarrow 2 = \frac{2}{3} + 2 + C \Rightarrow C = -\frac{2}{3}$

$f(x) = \frac{2}{3}x^{3/2} + 2x^{\frac{1}{2}} - \frac{2}{3} \Rightarrow f(4) = \underline{\underline{\frac{26}{3}}}$

2). Let $y = vx \Rightarrow y' = v'x + v$

$v'x + v = \dfrac{x^2 + v^2x^2}{vx^2 - x^2} = \dfrac{1+v^2}{v-1}$

$v'x = \dfrac{1+v^2}{v-1} - v = \dfrac{1+v^2 - v^2 + v}{v-1} = \dfrac{1+v}{v-1}$

$\dfrac{v-1}{v+1}\,dv = \dfrac{dx}{x} \Rightarrow \dfrac{v+1-2}{v+1}\,dv = \dfrac{dx}{x} \Rightarrow \left(1 - \dfrac{2}{v+1}\right)dv = \dfrac{dx}{x}$

$\therefore v - 2\ln|(v+1)| = \ln|x| + C$

$\therefore v = \ln|x| + 2\ln|v+1| + C \Rightarrow e^v = Ax(v+1)^2$

$\therefore e^{y/x} = Ax\left(\frac{y}{x} + 1\right)^2$

$y(1) = 0 \Rightarrow 1 = A \Rightarrow e^{y/x} = x\left(\frac{y}{x} + 1\right)^2$

$y(2) = a \Rightarrow e^{a/2} = 2\left(\frac{a}{2} + 1\right)^2 = \frac{a^2}{2} + 2a + 2$

$\Rightarrow \underline{\underline{a^2 + 4a + 4 - 2e^{\frac{a}{2}} = 0}}$

3). $\frac{1}{x}y' - 2y = \frac{1}{x}e^{x^2} \Rightarrow y' - 2xy = e^{x^2}$

Integrating factor $= e^{\int -2x\,dx} = e^{-x^2}$

$y = e^{x^2}\int e^{-x^2}e^{x^2}\,dx = e^{x^2}(x+c)$

$y(1) = 2e \Rightarrow 2e = e(1+c) \Rightarrow c = 1$

$\therefore y = xe^{x^2} + e^{x^2}$

$y(2) = 2e^4 + e^4 = \underline{\underline{3e^4}}$

4). $y' - \frac{1}{2x}y = -2x^3/2xy = -x^2y^{-1}$

Let $z = y^{1-(-1)} = y^2 \Rightarrow dz = 2y\,dy$

$\Rightarrow \frac{dz}{2y\,dx} - \frac{1}{2x}y = -\frac{x^2}{y}$

$\Rightarrow \frac{dz}{dx} - \frac{1}{x}y^2 = -2x^2 \Rightarrow \frac{dz}{dx} - \frac{1}{x}z = -2x^2$

Integrating factor $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$\therefore z = x\int \frac{1}{x}(-2x^2)\,dx = x(-x^2 + c)$

$\therefore y^2 = -x^3 + cx$

$y(1) = 2 \Rightarrow 4 = -1 + c \Rightarrow c = 5$

$y(\frac{1}{2}) = a \Rightarrow a^2 = -\frac{1}{8} + \frac{5}{2} = \underline{\underline{\frac{19}{8}}}$

5). $\frac{dT}{dt} = k(T-70) \Rightarrow T-70 = Ae^{kt}$

$T(0) = x \Rightarrow x-70 = A \Rightarrow T-70 = (x-70)e^{kt}$

$T(1) = 170 \Rightarrow 100 = (x-70)e^{kt} \Rightarrow 100^3 = (x-70)^3 e^{3k}$

$T(3) = 80 \Rightarrow 10 = (x-70)e^{3k}$

$\therefore \frac{100^3}{10} = (x-70)^2 \Rightarrow x = 70 + \sqrt{100000} \simeq \underline{\underline{386}}$

6). Let X = amount of substance at time t.

Let $X(0) = a$

$\frac{dx}{dt} = kx \Rightarrow x = Ae^{kt}$

$x(0) = a \Rightarrow a = A$

$\therefore x = ae^{kt}$

$x(100) = 0.9a \Rightarrow 0.9a = ae^{100k}$

$\Rightarrow \ln(0.9) = 100k$

$x(T) = 0.5a \Rightarrow 0.5a = ae^{Tk}$

$\Rightarrow \ln(0.5) = Tk$

$\therefore T = \frac{100\ln(0.5)}{\ln(0.9)} \simeq \underline{\underline{658}}$

7). $y'' - 0.01y = 0$

$\lambda^2 - 0.01 = 0 \Rightarrow \lambda = \pm 0.1$

$\therefore \underline{\underline{y = Ae^{0.1x} + Be^{-0.1x}}}$

8). $2y'' + 13y' + 15y = 0$

$\Rightarrow 2\lambda^2 + 13\lambda + 15 = 0$

$\Rightarrow (\lambda + 5)(2\lambda + 3) = 0$

$\Rightarrow \lambda = -5, -\frac{3}{2}$

$\therefore y = Ae^{-5t} + Be^{-\frac{3}{2}t}$

$\therefore \lim_{t \to \infty} y = \underline{\underline{0}}$

9). $y'' - 3y' + 2y = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2.$

For $y'' - 3y' + 2y = xe^x$:

let $y = ue^x$

$y' = u'e^x + ue^x$

$y'' = u''e^x + 2u'e^x + ue^x$

$\therefore u''e^x + 2u'e^x + ue^x - 3u'e^x - 3ue^x + 2ue^x = xe^x$

$\therefore u'' - u' = x$

let $u = Ax^2 + Bx + C$

$u' = 2Ax + B$

$u'' = 2A$

$\therefore 2A - 2Ax - B = x \Rightarrow A = -\frac{1}{2}, B = -1$

$\therefore$ General solution of $y'' - 3y' + 2y = xe^x$ is

$y = Ae^x + Be^{2x} + (-\frac{1}{2}x^2 - x)e^x.$

$\therefore y' = Ae^x + 2Be^{2x} + (-x-1)e^x + (-\frac{1}{2}x^2 - x)e^x$

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow A + B = 0 \\ y'(0) = -2 \Rightarrow A + 2B - 1 = -2 \end{array}\right\} \Rightarrow A = 1, B = -1.$$

$\therefore y = e^x - e^{2x} + (-\frac{1}{2}x^2 - x)e^x.$

$\therefore y(1) = e - e^2 + (-\frac{1}{2} - 1)e = \underline{\underline{-e^2 - \frac{1}{2}e}}$

10). $y''+2y'+y=0 \Rightarrow \lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1$ double root

For $y''+2y'+y=2+e^{2x}$ ---- ①

let $y=A+Be^{2x}$

$y'=2Be^{2x}$

$y''=4Be^{2x}$

$\therefore 4Be^{2x}+4Be^{2x}+A+Be^{2x}=2+e^{2x} \Rightarrow A=2,\ B=\frac{1}{9}$

$\therefore$ General solution of ① is

$y=Ae^{-x}+Bxe^{-x}+2+\frac{1}{9}e^{2x}$

$\therefore y'=-Ae^{-x}+Be^{-x}-Bxe^{-x}+\frac{2}{9}e^{2x}$

$y(0)=\frac{37}{9} \Rightarrow A+2+\frac{1}{9}=\frac{37}{9} \Rightarrow A=2$

$y'(0)=-\frac{7}{9} \Rightarrow -A+B+\frac{2}{9}=-\frac{7}{9} \Rightarrow B=A-1=1$

$\therefore y=2e^{-x}+xe^{-x}+2+\frac{1}{9}e^{2x}$

$\therefore y(-1)=2e-e+2+\frac{1}{9}e^{-2}$

$=e+2+\frac{1}{9e^2}$