NATIONAL UNIVERSITY OF SINGAPORE

MA1102R - CALCULUS

(Semester 2 : AY2013/14)

Time allowed : 2 hours

INSTRUCTIONS TO STUDENTS

1. This examination paper contains **FIVE** questions and comprises **THREE** printed pages.

2. Students are required to answer **ALL** questions. The maximum score for this examination is **70 Marks**.

3. Students should write the answers for each question on a new page.

4. This is a CLOSED BOOK (with helpsheet) examination.

5. Students are allowed to use two handwritten A4 size helpsheets.

6. Students may use scientific calculators. However, they should lay out systematically the various steps in the calculations.

Answer all the questions

Question 1 [15 marks]

(a) Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 0} \frac{x^2}{\cos x - 1}$.

(ii) $\displaystyle\lim_{x\to\infty} \frac{(3x^2+5)\sin(\frac{2}{x})}{5x+3}$.

(iii) $\displaystyle\lim_{x\to\ 0} \left(\frac{1+2^x+3^x}{3}\right)^{1/x}$.

(b) Use the $\epsilon - \delta$ definition of limit to prove that $\displaystyle\lim_{x\to -1} \frac{x^2-2x}{x+2} = 3$.

Question 2 [15 marks]

(a) Let $|f(x)|$ be a function differentiable at $x = 0$. If $f(0) = 0$, evaluate $f'(0)$. Justify your answer.

(b) Let $f(x)$ be a function continuous at $x = 0$. If $\displaystyle\lim_{x\to 0} \frac{f(x)}{x} = 2$, find the values of $f(0)$ and $f'(0)$. Justify your answer.

(c) Find the derivative of $\displaystyle\int_{\cos x}^{5x} \sin(t^2)\,dt$ with respect to x.

Question 3 [15 marks]

(a) Let $f(x)$ be the function defined by

$$f(x) = \begin{cases} x\cos(\frac{\pi x}{2}), & \text{if } x < 1, \\ ax^2 + b, & \text{if } x \geq 1. \end{cases}$$

If $f(x)$ is differentiable at $x = 1$, find the values of a and b.

(b) Let $f(x)$ and $g(x)$ be two functions such that $f(x) = (x-2)g(x)$ for all $x > 0$. If $g(x)$ is continuous at $x = 2$ and $g(2) = 1$, find the derivative of $f(x)$ at $x = 2$.

(c) By using the substitution $x = \dfrac{\pi}{4} - t$ or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{4}} \ln(\sqrt{1+\tan x})\,dx.$$

Question 4 [15 marks]

(a) Let $f(x)$ be a twice differentiable function on $(-\infty, \infty)$ such that its second derivative $f''(x)$ is continuous on $(-\infty, \infty)$. Suppose that $f(0) = f(1) = 0$ and $\int_0^1 f(x)\, dx = 1$.

(i) Evaluate $\int_0^1 x(1-x)f''(x)\, dx$. Justify your answer.

(ii) Prove that there exists a real number $c \in [0, 1]$ such that

$$\int_0^1 f(x)\, dx = -\frac{f''(c)}{12}.$$

(b) Let $f(x)$ be a continuous function on $[0, 1]$ such that $\int_0^1 f(x)\, dx = 0$ and $\int_0^1 xf(x)\, dx = 0$. Prove that there exist two points a and b in $[0, 1]$ such that $f(a) = f(b) = 0$.

(c) The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Question 5 [10 marks]

(a) By using the substitution $y = vx$, or otherwise, solve the initial value problem

$$(x^2 - xy)\frac{dy}{dx} + x^2 + y^2 = 0, \quad x > 0, \quad y(1) = 0.$$

(b) Solve the initial value problem

$$\frac{dy}{dx} - (1 + \frac{3}{x})y = x + 2, \qquad y(1) = e - 1, \qquad x > 0.$$

End of Paper