NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2013-2014

MA1101R Linear Algebra I

May 2014 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/registration number only. **Do not write your name.**
2. This examination paper contains a total of **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
3. Answer **ALL** questions. Each question carries 25 marks.
4. This is a **CLOSED BOOK** examination. **(For Non-H3 students:)** You are allowed to bring in two A4-sized help sheets which must be handwritten. Both sides of the A4-sized paper can be used. **(For H3 students:)** You are not allowed to bring in any help sheets. Formula sheets will be provided for you instead.
5. Calculators may be used. However, you should lay out systematically the various steps in the calculations

Question 1

(a) Consider the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 3 \\ -2 & 2 & 1 \end{pmatrix}.$$

(i) Find the inverse of $\boldsymbol{A}$.

(ii) Hence or otherwise, solve the following linear system:

$$\begin{cases} x - 2y - z = 2 \\ 2x + y + 3z = 5 \\ -2x + 2y + z = -5 \end{cases}$$

(iii) Find **four** elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}, \boldsymbol{E_4}$ such that

$$\boldsymbol{E_4 E_3 E_2 E_1 A}$$

is a matrix in row-echelon form.

(b) Let

$$\boldsymbol{u_1} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

(i) Show that $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbb{R}^3$.

(ii) Let $\boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Find $(\boldsymbol{w})_S$.

(iii) Let T be a linear operator on $\mathbb{R}^3$ such that

$$T(\boldsymbol{u_1}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad T(\boldsymbol{u_2}) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad T(\boldsymbol{u_3}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Find the standard matrix for T.

Hint: You may assume that $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ -2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 7 & -4 & 5 \\ -3 & 2 & -2 \end{pmatrix}$.

Question 2

(a) Consider the following linear system:

$$\begin{cases} x + y = 4 \\ 2x + y = 4 \\ 3x + y = 6 \\ 4x + y = 10. \end{cases}$$

(i) Find a least squares solution to the linear system.

(ii) Use your answer in (i) to find the projection of $\begin{pmatrix} 4 \\ 4 \\ 6 \\ 10 \end{pmatrix}$ onto the column space of $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$.

(iii) In general, suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is an inconsistent linear system. Prove that for all $k \neq 0$, $k \in \mathbb{R}$, the linear system $\boldsymbol{Ax} = k\boldsymbol{b}$ is also inconsistent. If $\boldsymbol{v}$ is a least squares solution for $\boldsymbol{Ax} = \boldsymbol{b}$, is $k\boldsymbol{v}$ a least squares solution for $\boldsymbol{Ax} = k\boldsymbol{b}$? Justify your answer.

(b) Let $W = \{(a, b, c, d, e) \mid a = 2b, c = d - 2e, a + 2d - e = 0\}$.

(i) Show that W is a subspace of $\mathbb{R}^5$.

(ii) Find a basis and determine the dimension of W.

(iii) Find a subspace V of $\mathbb{R}^5$ with dimension 3 such that $W + V = \mathbb{R}^5$. Justify your answer. (Express you answer for V in terms of a linear span.)

Question 3

(a) What is the condition that must be satisfied by a, b and c such that the linear system below is consistent?

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 \qquad\quad + x_3 = b \\ 2x_1 + x_2 + 3x_3 = c. \end{cases}$$

(b) Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$, where

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v_3} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

(i) You may assume that S is a basis for span(S). Using Gram-Schmidt Process and S, find an orthogonal basis for span(S).

(ii) Extend the basis obtained in (i) to an orthogonal basis for $\mathbb{R}^4$.

(c) Using the method of matrix diagonalization, solve the following recurrence relation. (**Note:** No marks will be given if you solve by other methods.)

$$a_n = \frac{1}{2}a_{n-1} + \frac{1}{2}a_{n-2} \quad \text{with } a_0 = 0,\ a_1 = 1.$$

Hint: You do not need to compute $\boldsymbol{P}$ explicitly.

Question 4

(a) Let $\boldsymbol{A}$ be the following matrix:

$$\begin{pmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis for the row space of $\boldsymbol{A}$.

(ii) How many solutions does $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$ have? Justify your answer.

(iii) If $\boldsymbol{A}$ is the standard matrix for a linear transformation T, find a basis for the kernel of T and determine nullity(T).

(b) Let $\boldsymbol{B}$ be the following matrix:

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}.$$

(i) Show that the characteristic equation of $\boldsymbol{B}$ is $\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$. It is known that -1 is one of the eigenvalues of $\boldsymbol{B}$. Find all other eigenvalues of $\boldsymbol{B}$.

(ii) For each eigenvalue λ of $\boldsymbol{B}$, find a basis for the eigenspace E_λ.

(iii) Is $\boldsymbol{B}$ diagonalizable? Justify your answer. If $\boldsymbol{B}$ is diagonalizable, write down a matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$ is a diagonal matrix $\boldsymbol{D}$. Write down the matrix $\boldsymbol{D}$ corresponding to your choice of $\boldsymbol{P}$.

(iv) Determine if the following matrix is diagonalizable. Justify your answer.

$$\begin{pmatrix} 2016 & 2 & 3 \\ 1 & 2016 & 1 \\ 2 & -2 & 2015 \end{pmatrix}.$$

END OF PAPER