NATIONAL UNIVERSITY OF SINGAPORE

MA1102R CALCULUS

SEMESTER 1 EXAMINATION 2013 – 2014

Examiners: Prof. Goh Say Song & Dr. Wang Fei

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation number only. Do not write your name.
2. This examination paper contains a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.
3. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
4. Use a separate page for each question.
5. This is a **CLOSED BOOK** examination.
6. Two pieces of A4-sized formula sheet are prepared and provided by the examiners.
7. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [8 marks]

Let $f(x) = 2x^6 - 15x^4 + 4$.

(i) Find the open intervals on which f is increasing and decreasing.

(ii) Find the coordinates of all the local maximum and minimum points of f.

(iii) Find the open intervals on which f is concave up and concave down.

(iv) Find the coordinates of all the inflection points of f.

Question 2 [11 marks]

(a) Find the limit $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{x^2}{x-1}\right)$.

(b) Use the ϵ, δ-definition to prove that $\lim_{x\to -1}\frac{2-x}{1+x^2} = \frac{3}{2}$.

Question 3 [14 marks]

Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2}{(x^2-3x+2)^2}\,dx$.

(b) $\displaystyle\int_0^1 \frac{1}{(2-x)\sqrt{1-x}}\,dx$.

Question 4 [8 marks]

(i) Show that for any positive, continuous function f on the interval $[5,7]$,

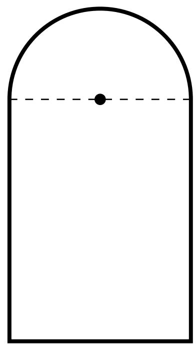
$$\int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}\,dx = \int_2^4 \frac{f(x+3)}{f(9-x)+f(x+3)}\,dx.$$

(ii) Using the result in part (i), evaluate

$$\int_2^4 \frac{\sqrt[5]{9-x}}{\sqrt[5]{9-x}+\sqrt[5]{x+3}}\,dx.$$

Question 5 [8 marks]

A Norman window has the shape of a rectangle surmounted by a semicircle, as indicated in the figure.



Find the dimensions of a Norman window of perimeter 9 meters that will have the largest area.

Question 6 [13 marks]

(a) Find the volume of the solid generated by revolving the region enclosed by the curve $y = x + \dfrac{4}{x}$ and the line $y = 5$ about the line $x = -1$.

(b) Find the length of the curve $y = 8\left(\ln \dfrac{2+\sqrt{x}}{2-\sqrt{x}} - \sqrt{x}\right)$ from $x = 0$ to $x = 1$.

Question 7 [16 marks]

(a) Let f be a continuous function on $\mathbb{R}$. Define

$$F(x) = \int_0^x f(t)(x-t)^2\, dt.$$

Evaluate $F'''(x)$.

(b) Let $0 < a < b$. Evaluate

$$\lim_{t \to 0} \left\{ \int_0^1 [a(1-x) + bx]^t\, dx \right\}^{\frac{1}{t}}.$$

(c) Show that if f is differentiable at a, then for any number $n \neq 0, 1$,

$$\lim_{h \to 0} \frac{f(a+nh) - f(a+(n-1)h)}{h} = f'(a).$$

Question 8 [16 marks]

(a) Solve the differential equation
$$\frac{dy}{dx} = (y + e^{-x}) \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$
given the initial condition that $y = 1$ when $x = 0$.

(b) The temperature in an air-conditioned room is 20°C. A thermometer which has been kept in it is placed outside. In 5 minutes the thermometer reading is 25°C. Another 5 minutes later, it becomes 28°C. Let T be the reading when the thermometer is placed outside for t minutes. It is known that T satisfies the differential equation
$$\frac{dT}{dt} = -k(T - T_S),$$
where T_S is the outdoor temperature, and k is a positive constant. Find the outdoor temperature.

Question 9 [6 marks]

Suppose that f is differentiable on $\mathbb{R}$ and
$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0.$$
Prove that there exists $c \in \mathbb{R}$ such that $f'(c) = 0$.

[End of Paper]