NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2013-2014

MA1101R Linear Algebra I

November 2013 — Time allowed : 2 hours

Examiner: Dr. Ng Kah Loon

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/registration number only. **Do not write your name.**
2. This examination paper contains a total of **FOUR (4)** questions and comprises **THREE (3)** printed pages.
3. Answer **ALL** questions. Each question carries 25 marks.
4. This is a **CLOSED BOOK** examination. You are allowed to bring in two A4-sized help sheets which must be handwritten. Both sides of the A4-sized paper can be used.
5. Non-graphing, non-programmable calculators may be used. However, you should lay out systematically the various steps in the calculations

Question 1

Let

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{b_1} = \begin{pmatrix} 3 \\ -3 \\ 3 \\ -3 \end{pmatrix}, \quad \boldsymbol{b_2} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

(i) Show that $\boldsymbol{v} = (1, 1, 2, 2)^T$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b_1}$.

(ii) Find a basis for the nullspace of $\boldsymbol{A}$.

(iii) Using your answers in (i) and (ii), write down the solution set for $\boldsymbol{Ax} = \boldsymbol{b_1}$.

(iv) Show that $\boldsymbol{Ax} = \boldsymbol{b_2}$ is inconsistent.

(v) Find **two different** least squares solutions $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, for the linear system $\boldsymbol{Ax} = \boldsymbol{b_2}$. Verify that $\boldsymbol{Ax_1} = \boldsymbol{Ax_2}$.

(vi) If $\boldsymbol{b_3} \in \mathbb{R}^4$ **does not** belong to the column space of $\boldsymbol{A}$, is it possible that $\boldsymbol{Ax} = \boldsymbol{b_1} + \boldsymbol{b_3}$ is consistent? Justify your answer.

Question 2

Let

$$\boldsymbol{X} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}.$$

(i) Find $\boldsymbol{X}^{-1}$ by Gaussian Elimination. (You may assume that $\boldsymbol{X}^{-1}$ exists.)

(ii) Find $\det(\boldsymbol{X})$ and thus write down the matrix $\mathbf{adj}(\boldsymbol{X})$.

(iii) If $\boldsymbol{X}$ is the standard matrix for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, find the formula of the transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T \circ S$ satisfies $(T \circ S)(\boldsymbol{x}) = \boldsymbol{x}$ for all nonzero vectors $\boldsymbol{x} \in \mathbb{R}^3$.

(iv) Find all eigenvalues λ of $\boldsymbol{X}$ and for each λ, find a basis for E_λ, the eigenspace of $\boldsymbol{X}$ associated with λ.

(v) Is $\boldsymbol{X}$ diagonalizable? If it is, find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{X}$.

(vi) Let $\boldsymbol{Y}$ be a diagonalizable matrix of order n. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of $\boldsymbol{Y}$ with eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$ and $S_{\lambda_1}, S_{\lambda_2}, \ldots, S_{\lambda_k}$ are the corresponding bases for $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$. Prove that

$$E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k} = \mathbb{R}^n.$$

(**Remark:** For two subspaces V and W of $\mathbb{R}^n$, $V + W$ is defined as

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V, \boldsymbol{w} \in W\}.)$$

Question 3
Let

$$\boldsymbol{u_1} = (1,1,0,2), \quad \boldsymbol{u_2} = (0,-2,1,1), \quad \boldsymbol{u_3} = (0,1,1,2)$$
$$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}, \quad V = \text{span}(S).$$

(i) Prove that S is a basis for V.

(ii) Let $\boldsymbol{w} = (3,-1,-1,3)$. Determine if $\boldsymbol{w}$ belongs to V. If it does, find $(\boldsymbol{w})_S$.

(iii) Apply Gram-Schmidt Process to S to obtain an orthogonal basis T for V.

(iv) Find $(\boldsymbol{w})_T$.

Hint: You may use the following:

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{6} & 0 \\ 1 & -2 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{5}{6} & 1 \\ 2 & 1 & \frac{1}{6} & 2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{6} \\ 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

(v) Extend S to a basis for $\mathbb{R}^4$.

(vi) Find a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ such that the kernel of T is V.

Hint: You may give your answer by finding the standard matrix of T.

Question 4 Let

$$\boldsymbol{Y} = \begin{pmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{pmatrix}, \quad \boldsymbol{Z} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{v_1} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix} 1 \\ 5 \\ -1 \\ 3 \end{pmatrix}.$$

(i) Show that $\text{rank}(\boldsymbol{Y}) = 2$.

(ii) Determine $\text{nullity}(\boldsymbol{Y})$. Hence show that $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is a basis for the nullspace of $\boldsymbol{Y}$.

(iii) Determine if $\boldsymbol{Zx} = \boldsymbol{v_1}$ is consistent. If it is, express $\boldsymbol{v_1}$ as a linear combination of the columns of $\boldsymbol{Z}$. If not, find a vector $\boldsymbol{w}$ such that $\boldsymbol{Zx} = \boldsymbol{w}$ is consistent and $d(\boldsymbol{w}, \boldsymbol{v_1})$ is as small as possible.

(iv) Are there elementary matrices $\boldsymbol{E_1}, \ldots, \boldsymbol{E_k}$ such that $\boldsymbol{E_k} \cdots \boldsymbol{E_1 Y} = \boldsymbol{Z}$? If so, find such $\boldsymbol{E_1}, \ldots, \boldsymbol{E_k}$. If not, explain why.

(v) Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ be any 4 vectors in $\mathbb{R}^4$ and $U = \{\boldsymbol{Yu_1}, \boldsymbol{Yu_2}, \boldsymbol{Yu_3}, \boldsymbol{Yu_4}\}$. Determine the largest possible value of $\dim(\text{span}(U))$.

END OF PAPER