NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2012-2013

MA3227 Numerical Analysis II

April 2013 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **Four (4)** printed pages.
2. Answer **ALL** questions.
3. This is a closed book exam. However, candidates are allowed to bring an A4 sized help sheet which can be written on both sides.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Suppose $B = uv^\top$, where $u, v \in \mathbb{R}^{n\times 1}$. Prove that $\|B\|_2 = \|u\|_2\|v\|_2$.

(b) Let $n \geq 2$. Prove that $\|I - uu^\top\|_2 = 1$ where $I \in \mathbb{R}^{n\times n}$ is the identity matrix and $u \in \mathbb{R}^{n\times 1}$ satisfies $\|u\|_2 = 1$. [Hint: Recall $\|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$. If $\{u_i, i = 1, ..., n\}$ is an orthonormal basis of $\mathbb{R}^{n\times 1}$, any vector $y \in \mathbb{R}^{n\times 1}$ can be written as $\sum_{i=1}^n \beta_i u_i$ for some $\beta_i \in \mathbb{R}$. How are $\|y\|_2$ and $\sqrt{\sum_{i=1}^n \beta_i^2}$ related? In the proof, you can also use the fact that for any $u \in \mathbb{R}^{n\times 1}$ with $\|u\|_2 = 1$, you can extend it to $\{u, u_2, u_3, ..., u_n\}$ which form an orthonormal basis.]

Question 2 [20 marks]

Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Suppose there is a $B \in \mathbb{R}^{n\times n}$ whose inverse is easy to calculate, and suppose $B + B^\top - A$ is also positive definite. Consider the following iteration for solving $Ax = b$:

$$x_{k+1} = x_k + B^{-1}(b - Ax_k). \tag{1}$$

Prove that the iteration will converge for any initial guess x_0 by going through the following steps:

(a) Let $e_k = x - x_k$. Find a matrix H so that $e_{k+1} = He_k$. [5 marks]

(b) Let $\lambda \in \mathbb{C}$ be any eigenvalue of H. Denote its associated eigenvector by u. Note that u in general is a complex vector in $\mathbb{C}^{n\times 1}$. Find a function $g(\lambda)$ so that

$$u^*(B + B^\top - A)u = g(\lambda)\, u^*Au,$$

where u^* is the conjugate transpose of u. [10 marks] [Hint: First derive an equation for u^*Au, and then take the conjugate transpose of it.]

(c) Prove that $|\lambda| < 1$ and hence the spectral radius of H is strictly less than one. [5 marks]

Question 3 [20 marks]

(a) Perform one iteration of Newton's method for solving

$$\begin{cases} xy^2 + x^2y + x^4 - 3 &= 0 \\ x^3y^5 - 2x^5y - x^2 + 3 &= 0\,, \end{cases}$$

starting with initial guess $(1,1)^\top$.

(b) Prove that if f is continuous on $[a,b]$ and satisfies $a \le f(a)$ and $f(b) \le b$, then f has a fixed point in the interval $[a,b]$. Note that we do not assume $a \le f(x) \le b$ for all x in $[a,b]$. [Hint: What is the theorem that lies behind the bisection method for finding a root of $g(x)$?]

Question 4 [25 marks]

Suppose you want to approximate $I = \int_0^1 f(x)dx$ by Monte Carlo method, where $f(x) = \frac{1}{x^{1/3}} + \frac{x}{10}$.

(a) If you use simple sampling to approximate I by $\hat{I}_n = \frac{1}{n}\sum_{i=1}^n f(U_i)$, where $U_1, ..., U_n$ are i.i.d. $U(0,1)$ random variables, what is the variance of $\hat{I}_n$? How is the variance of $\hat{I}_n$ related to $|\hat{I}_n - I|$? [10 marks]

(b) How do you generate a random variable X with given pdf $g(x) = \frac{2}{3x^{1/3}}$? [5 marks]

(c) If you use importance sampling based on the pdf $g(x) = \frac{2}{3x^{1/3}}$, what is the new formula of $\hat{I}_n$ once you have generated n i.i.d. random variables $X_1, X_2, ..., X_n$ whose common pdf is $g(x)$? What is the variance of this new $\hat{I}_n$? [10 marks]

Question 5 [15 marks]

Suppose we have m items. The ith item is worth v_i dollars and weighs w_i kilograms. We can put some of these items into a "knapsack", but the total weight cannot be more than b kilograms. We like to find out what is the most valuable subset of items that will fit into the knapsack. To formulate it as a mathematical problem, let $\vec{w} = (w_1, ..., w_m) \in \mathbb{R}^m$, $\vec{v} = (v_1, ..., v_m) \in \mathbb{R}^m$, and $\vec{z} = (z_1, ..., z_m) \in \{0,1\}^m$, where $z_i = 1$ if item i is put into the knapsack, and $z_i = 0$ otherwise. $\vec{z}$ is called the decision vector. Clearly, $\vec{w} \cdot \vec{z} = \sum_{i=1}^m w_i z_i$ is the total weight and $\vec{v} \cdot \vec{z} = \sum_{i=1}^m v_i z_i$ is the total value. The set of all permissible ways to pack the knapsack is $S = \{\vec{z} \in \{0,1\}^m : \vec{w} \cdot \vec{z} \leq b\}$. We can now express our original optimization problem as the following integer programming problem:

$$\text{maximize } \vec{v} \cdot \vec{z} \quad \text{subject to } \vec{z} \in S.$$

We would like to generate a Markov chain $\{X_1, X_2, ..., X_n, ...\}$ on S. After a large number of X_i's have been generated, we then choose the one for which the total value $\vec{v} \cdot X_i$ is largest.

We can construct a S-valued Markov chain by the following algorithm.

Given the current $X_t = \vec{z} = (z_1, ..., z_m) \in S$:

1) Choose $J \in \{1, ..., m\}$ uniformly at random.
2) Flip z_J: that is, let $\vec{y} = (z_1, ..., z_{J-1}, 1 - z_J, z_{J+1}, ..., z_m)$.
3) If $\vec{y} \in S$, then set $X_{t+1} = \vec{y}$. If $\vec{y} \notin S$, then set $X_{t+1} = \vec{z}$.

Now, answer the following three questions:

(a) In the above Markov chain, if both $\vec{z}$ and $\vec{y}$ are in S, and if $\vec{z}$ can be changed to $\vec{y}$ by flipping just one component of $\vec{z}$, what is the transition probability $p_{\vec{z}\vec{y}}$ and $p_{\vec{y}\vec{z}}$? [5 marks]

(b) What is the invariant distribution of the above Markov chain on S? [5 marks]

(c) If we want to sample from S so as to make better solutions more likely, we may want the invariant distribution to be $\pi_\beta(\vec{z}) = C^{-1} e^{\beta(\vec{v} \cdot \vec{z})}$ for some given parameter $\beta > 0$ and some unknown constant $C > 0$. If so, what is the Metropolis algorithm for generating such a Markov chain? [5 marks]

END OF PAPER