NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER II EXAMINATION 2012-2013

# MA2214 Combinatorial Analysis

April/May 2013— Time allowed : 2 hours

---

**INSTRUCTIONS TO CANDIDATES**

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **EIGHTEEN (18)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. For questions in this section, write down the numerical (no formulas) answer in the answer booklet. You may use the first page of the answer booklet to write down **ALL** the answers in this section. **NO** working is required. Each question in this section carries **2** marks.
3. Answer **ALL** questions in **Section B**. For questions in this section, show your working and write down your explanations **clearly**. Failure to do so will result in marks deducted. Start on a new page for each question in this section. Each question in this section carries **10** marks.
4. Calculators may be used. However, you should lay out systematically the various steps in the calculations

**Section A**

1 There are 10 identical blue bottles and 5 identical red bottles. How many ways are there to arrange the 15 bottles in a row such that no two red bottles are next to each other?

2 Find the number of 17-digit binary sequences with more 0's than 1's.

3 I wish to distribute 10 distinct toys to my 3 children Amy, Bob and Charlie. Each child must get at least one toy but Amy must receive an even number of toys while Charlie must receive an odd number. How many ways can I go about my distribution?

4 How many ways are there to distribute 12 identical pencils to 3 students if each student gets at least one pencil but no more than five pencils?

5 A basket contains 10 apples, 11 bananas and 12 pears. What is the smallest number of fruits that one must pick to ensure that there are always 7 pieces of the same type?

6 Ten lucky contestants in a radio talkshow have won themselves each a prize. A total of 12 items, comprising of 4 identical hampers, 4 identical gift vouchers and 4 identical digital cameras are available as prizes. How many ways can the ten contestants be awarded their prizes?

7 5 couples (husband and wife) and 5 single women are at a party. By choosing from the 10 women at the party, how many ways are there are to assign the 5 men a dancing partner each such that exactly two men are dancing with their wives?

8 6 boys $(B_1, \cdots, B_6)$ and 4 girls $(G_1, \cdots, G_4)$ are to be seated around a table. How many ways can they be seated if $G_1$ **cannot** have $B_1$ and $B_2$ by her side.

9 Ten customers arrive at a bank at different times, waiting to be served by a customer service officer. They are served, one at a time, by the officer in a random order. A person is said to be *unlucky* if that person was the $i$th to arrive but is not among the first $i$ people served by the officer. $X$, $Y$, $Z$ are three of the customers that arrived third, fifth and eighth respectively. In how many of the 10! possible random orders will at least one of the customers be unlucky?

10 How many even integers are there between 2000 and 5000 such that no digit is repeated?

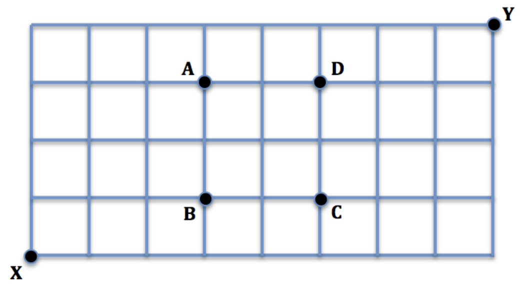
**Section B**

11 (i) Prove that there are 325 integers that can be formed using the numbers $1, 2, 3, 4, 5$ without repetition (for example, 4, 52, 123, 4513 and 13254 are some of these integers).

(ii) If we list all the 325 integers in ascending order (smallest to largest), what position does the integer 2345 occupy?

(iii) What is the 300th integer in this ascending order?

12 A person wishes to start at point **X** and reach point **Y** by making the smallest number of moves required.




(i) How many different ways can this be done if he wishes to avoid all the points **A,B,C,D**?

(ii) How many different ways can this be done if he wishes to pass through exactly two of the four points **A,B,C,D**?

13 (i) Show that for every natural number $k$, $10^k$ and 2013 are relatively prime.

(ii) Prove that there is a number in the set $S = \{5, 55, 555, \cdots, \underbrace{555\ldots5}_{2013}\}$ that is divisible by 2013.

14 Find, in closed form, an ordinary generating function for the sequence $(a_n)$ where $a_n$ denotes

(i) $n$.

(ii) $\binom{3+n}{n}$.

(iii) $\sum_{r=1}^{n}(n+1-r)r$.

15 Let $S = \{1, 2, 3, \cdots, n+1\}$ where $n \geq 3$, and let

$$T = \{(w, x, y, z) \in S^4 \mid w, x, y < z\}.$$

By counting $|T|$ in two different ways, show that

$$\sum_{r=1}^{n} r^3 = \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4}.$$

16 (a) Give a combinatorial proof for the following identity.

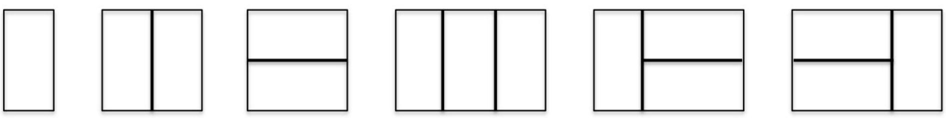
$$\binom{k}{2} + \binom{n-k}{2} + k(n-k) = \binom{n}{2}, \quad \text{for all } k \leq n.$$

(b) Prove that

$$\sum n_1 \binom{n}{n_1, n_2, \cdots, n_m} = nm^{n-1}$$

where the sum is taken over all $m$-ary sequences $(n_1, n_2, \cdots, n_m)$ of non negative integers with $\sum_{i=1}^{m} n_i = n$.

17 (a) Let $a_n$ be the number of ways to pave a $2 \times n$ rectangular floor with $1 \times 2$ tiles. All the ways of paving a $2 \times n$ rectangular floor where $n \leq 3$ are shown below. Obtain a recurrence relation for $a_n$ with two initial conditions (do not solve the recurrence relation).



(b) Let $b_n$ denote the number of ternary sequences of length $n$ where any 1 must be followed immediately by 0 (that is, the patterns '11' and '12' do not appear in the sequence). Find a recurrence relation for $b_n$ and solve it.

(*Please turn over for Question 18...*)

18 (a) Let $2n$ players participate in a badminton tournament and $a_n$ be the number of different pairs that can be formed for the $n$ matches in the first round.

(i) Find a recurrence relation for $a_n, n \geq 2$ and with initial condition $a_1 = 1$.

(ii) By solving the recurrence relation obtained in (i), show that

$$a_n = \frac{(2n)!}{2^n n!}, \quad n = 1, 2, \cdots .$$

(b) Let $k \in \mathbb{N}$. Using the method of generating functions, solve the recurrence relation

$$a_n - ka_{n-1} = k^n$$

given that $a_0 = 1$.

END OF PAPER