NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2012-2013

MA2108 Mathematical Analysis I

May 2013 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.

3. Answer **ALL** questions in **Section A**. Section A carries a total of 70 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Section B carries a total of 30 marks.

5. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 70 marks.*

Question 1.

Let

$$x_1 = 1, \quad x_{n+1} = \frac{\sqrt{8x_n^2 + 9}}{3}, \qquad n \in \mathbb{N}.$$

(i) Prove that $x_n \leq 3$ for all $n \in \mathbb{N}$. [3 marks]

(ii) Prove that (x_n) converges and find its limit. [7 marks]

Question 2.

(a) Test the following series for convergence.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)!}{(n!)^2\, 5^n}$. [4 marks]

(ii) $\displaystyle\sum_{n=1}^{\infty} n \left(1 + \frac{1}{4n}\right)^{-2n^2}$. [4 marks]

(b) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$. [5 marks]

(c) For each $n \in \mathbb{N}$, let

$$a_n = \begin{cases} \dfrac{1}{\sqrt{n}} & \text{if } n \text{ is odd} \\[2ex] \dfrac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \frac{1}{\sqrt{1}} - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{\sqrt{5}} - \frac{1}{6} + \frac{1}{\sqrt{7}} - \frac{1}{8} + \cdots$$

converges. Justify your answer. [7 marks]

Question 3.

(a) Use the $\varepsilon - \delta$ definition of limit to prove that

$$\lim_{x\to 1} \frac{x+1}{4x-3} = 2.$$

[7 marks]

(b) In each part, either evaluate the limit or show that the limit does not exist. Here $[x]$ denotes the greatest integer less than or equal to x.

(i) $\displaystyle\lim_{x\to 0} \cos\left(\frac{1}{x^2}\right).$ [4 marks]

(ii) $\displaystyle\lim_{x\to 0^+} x^3\left(\left[\frac{1}{x^3}\right] + \left[\frac{2}{x^3}\right]\right).$ [4 marks]

Question 4.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2+2 & \text{if } x \text{ is rational} \\ 4x-3 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not continuous at any point of $\mathbb{R}$. [7 marks]

(b) The functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous at the point $x = a$ and $g(a) > h(a)$. Prove that there exists $\delta > 0$ such that

$$g(x) > h(x) \qquad \text{for all } x \in (a-\delta, a+\delta).$$

[10 marks]

Question 5.

Determine whether the function

$$g(x) = (x-2)^2 \sin\left(\frac{x^2}{2-x}\right)$$

is uniformly continuous on $[1, 2)$. Justify your answer. [8 marks]

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 30 marks.*

Question 6.

(a) Prove that if (a_n) converges and (b_n) is a bounded sequence, then

$$\limsup\,(a_n + b_n) = \lim_{n\to\infty} a_n + \limsup\; b_n.$$

[8 marks]

(b) Let (x_n) be a sequence such that the series $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ converges. Prove that (x_n) converges.

[7 marks]

Question 7.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$ and has the property that

$$f(x) + f(2x) = 0 \quad \forall x \in \mathbb{R}.$$

(i) Find the value of $f(0)$. [3 marks]

(ii) Prove that f is a constant function. [5 marks]

(b) The function $h : (0,1) \to \mathbb{R}$ has the following properties:

(i) h is increasing on $(0,1)$.

(ii) The range of h is an open interval (a,b).

Prove that h is continuous on $(0,1)$. [7 marks]

Question 8.

(a) Suppose that the function $f : [0,\infty) \to \mathbb{R}$ is continuous on $[0,\infty)$ and

$$\lim_{x\to\infty} f(x) = 1.$$

Prove that f is bounded on $[0,\infty)$. [7 marks]

(b) Let the function $g : [1,\infty) \to \mathbb{R}$ be uniformly continuous on $[1,\infty)$. Prove that there exists $M > 0$ such that

$$|g(x)| \leq Mx \quad \text{for all } x \geq 1.$$

[8 marks]

END OF PAPER