NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2012-2013

MA2101 Linear Algebra II

April 2013 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections. It contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.

3. Answer not more than **TWO (2)** questions in **Section B**. Each question in Section B carries 20 marks.

4. Calculators can be used. However, various steps in the calculations should be laid out systematically.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 Marks]

(In this question, vectors in $\mathbb{R}^2$ are written as column vectors.)

Let $V = \mathcal{M}_{2\times 2}(\mathbb{R})$ and $W = \{\boldsymbol{A} \in V \mid \boldsymbol{Au} = \boldsymbol{0}\}$ where $\boldsymbol{u} = (1, -1)^{\mathrm{T}}$.

(a) Show that W is a subspace of V.

(b) Find a basis for W and determine the dimension of W.

(c) Give an example of a subspace W' of V such that $V = W \oplus W'$.

Question 2 [15 Marks]

Let $T : \mathcal{P}_2(\mathbb{C}) \to \mathbb{C}^3$ be a linear transformation such that $[T]_{E,B} = \begin{pmatrix} 1 & \mathrm{i} & 1 \\ 0 & 1 & \mathrm{i} \\ 1 & 2\mathrm{i} & 0 \end{pmatrix}$ where $B = \{1,\, x,\, x^2\}$, $E = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ and $\mathrm{i} = \sqrt{-1}$.

(a) Find a basis for $\mathrm{Ker}(T)$. Hence, or otherwise, find $\mathrm{nullity}(T)$ and $\mathrm{rank}(T)$.

(b) Let $C = \{1,\, 1 + \mathrm{i}x,\, 1 + x^2\}$. Compute $[T]_{E,C}$.

(c) Find an invertible matrix $\boldsymbol{P}$ such that $[T]_{E,B}\boldsymbol{P} = [T]_{E,C}$.

Question 3 [15 Marks]

Let T be a linear operator on $V = \mathcal{M}_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} b+d & a \\ 3c-d & 2c \end{pmatrix} \quad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V.$$

(a) Compute the characteristic polynomial of T.

(b) Find a real polynomial $p(x)$ so that $T^{-1} = p(T)$.

(c) Find the dimension of the eigenspace of T associated with the eigenvalue 1 and hence write down a Jordan canonical form for T.

Question 4 [15 Marks]

(a) Give an example of a real 2×2 matrix such that it is normal but not symmetric.

(b) Let $\boldsymbol{A}$ be a real normal matrix. Prove that if all eigenvalues of $\boldsymbol{A}$ are real numbers, then $\boldsymbol{A}$ is symmetric.

(c) Restate the result of (b) in terms of a linear operator.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 Marks]

Let S and T be linear operators on a finite dimensional vector space V.

(a) Describe how to find ordered bases B and C for V such that

$$[S]_{C,B} = \begin{pmatrix} \boldsymbol{I}_m & \boldsymbol{0}_{m\times n} \\ \boldsymbol{0}_{n\times m} & \boldsymbol{0}_{n\times n} \end{pmatrix}$$

where $m = \text{rank}(S)$ and $n = \text{nullity}(S)$. (You do not need to prove that B and C are bases for V.)

(b) Let $[T]_{B,C} = \begin{pmatrix} \boldsymbol{W} & \boldsymbol{X} \\ \boldsymbol{Y} & \boldsymbol{Z} \end{pmatrix}$ where $\boldsymbol{W}$, $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices respectively. Compute $[T \circ S]_B$ and $[S \circ T]_C$.

(c) Hence, or otherwise, prove that $T \circ S$ and $S \circ T$ have the same characteristic polynomials.

(d) Is it true that $T \circ S$ and $S \circ T$ always have the same minimal polynomials? Justify your answer.

Question 6 [20 Marks]

Let T be a linear operator on a finite dimensional vector space V. Suppose λ is an eigenvalue of T. Define $Q = T - \lambda I_V$ and $K_i = \mathrm{Ker}(Q^i)$ for $i = 1, 2, 3, \ldots$.

(a) Prove that

(i) $K_i \subseteq K_{i+1}$ for all $i \geq 1$; and

(ii) if $K_k = K_{k+1}$ for some k, then $K_k = K_m$ for all $m \geq k$.

(b) Suppose s is the smallest positive integer such that $K_s = K_{s+1}$. Let $K = K_s$ and $R = R(Q^s)$. It is known that K and R are T-invariant subspaces of V.

(i) Prove that $V = K \oplus R$.

(ii) Find the minimal polynomial of $T|_K$.

Question 7 [20 Marks]

Let T be a linear operator on an inner product space V. Suppose the adjoint T^* of T exists.

(a) Prove that $\mathrm{Ker}(T^* \circ T) = \mathrm{Ker}(T)$.

(b) Is it true that $\mathrm{Ker}(T \circ T^*) = \mathrm{Ker}(T)$? Justify your answer.

(c) Given $\boldsymbol{b} \in V$, show that $\boldsymbol{x} = \boldsymbol{u}$ is a solution to $(T^* \circ T)(\boldsymbol{x}) = T^*(\boldsymbol{b})$ if and only if $T(\boldsymbol{u})$ is the orthogonal projection of $\boldsymbol{b}$ onto $\mathrm{R}(T)$.

(d) Given $\boldsymbol{b} \in \mathrm{R}(T)$, show that $\{\boldsymbol{u} \mid T(\boldsymbol{u}) = \boldsymbol{b}\} = \{\boldsymbol{u} \mid (T^* \circ T)(\boldsymbol{u}) = T^*(\boldsymbol{b})\}$.

[END OF PAPER]