NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER II EXAMINATION 2012-2013

MA1104 Multivariable Calculus

May 2013 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-size help sheets into the examination room.

2. This examination paper contains a total of **ELEVEN (11)** questions and comprises **SIX (6)** printed pages.

3. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [13 marks]

Let $A(1,1,-1)$, $B(5,4,-3)$ $C(1,-2,0)$ and $D(7,9,-3)$ be four points.

(i) Let $ax+by+cz=1$ be the equation of the plane containing the triangle ABC. Determine a,b,c.

(ii) Compute the area of the triangle ABC.

(iii) Find the distance from D to the plane in (i).

(iv) Determine the point P on the plane in (i) which is nearest to the point D.

Warning. A mistake in (i) will give wrong answers in (ii), (iii) and (iv).

Question 2. [12 marks]

State clearly if each of the following limits exists:

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+4y^6}$.

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\cos(x)\sin^3(y)}{x^4+4y^6}$.

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|^\alpha|y|^\beta}{x^4+4y^6}$ where $\alpha=\frac{23}{10}$ and $\beta=\frac{27}{10}$.

If the limit does not exist, give a proof.
If the limit exists, compute its value.

Question 3. [7 marks]

Given $z = f(x, y)$ satisfies

$$xy^2z + z^3 - 2x + 7 = 0.$$

(i) Find the equation of the tangent plane at $P(1, 2, -1)$.
Express your answer in the form $z = ax + by + c$.

(ii) Approximate $f(1.021, 1.993)$.
Express your answer to 3 decimal places.

Question 4. [8 marks]

Let

$$f(x, y) = \sqrt{|xy|}.$$

(i) Does $\dfrac{\partial f}{\partial x}(0, 0)$ exist?
If it exists, compute its value.
If it does not exist, give a proof.

(ii) Is $f(x, y)$ a differentiable function at (0,0)?
Justify your answer.

Remark. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is a *differentiable* function at $(0, 0)$ if the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist, and there exist functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ such that

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + x\epsilon_1(x, y) + y\epsilon_2(x, y)$$

where $\displaystyle\lim_{(x,y)\to(0,0)} \epsilon_1(x, y) = \lim_{(x,y)\to(0,0)} \epsilon_2(x, y) = 0.$

Question 5. [10 marks]

Find all the local maximum points, local minimum points and saddle points of the graph

$$z = 16x^3 - 108xy^3 + 81y^6.$$

Question 6. [10 marks]

Given two surfaces $z = 6x + 8y$ and $z = x^2 + y^2$.

(i) Let C be the curve which is the intersection of the two surfaces.
Let
$$\mathbf{r}(t) = \langle 8, 4, 80\rangle + t\langle a, b, c\rangle.$$
be the equation of the tangent line of C at the point $P(8, 4, 80)$.
Determine a, b, c.

(ii) Compute the volume of the solid bounded by the two surfaces.

Question 7. [10 marks]

Let
$$D = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$$
denote the tetrahedron.

We consider the change of coordinates

$$\begin{aligned} x &= u - uv, \\ y &= uv - uvw, \\ z &= uvw. \end{aligned}$$

(i) Compute the Jacobian $J(u, v, w)$.

(ii) Let a, b, c be three non-negative integers. Using (i) or otherwise, compute the following triple integral in terms of a, b, c:
$$\iiint_D x^a y^b z^c dxdydz.$$

Hint: For (ii) you may assume that for a pair of positive integers m and n, the integral
$$\int_0^1 t^m(1-t)^n dt = \frac{m!n!}{(m+n+1)!}$$
where $m! = 1 \cdot 2 \cdot 3 \cdots m$.

Question 8. [5 marks]

Let D be a domain in the first quadrant in the xy-plane.

We suppose D has density 1. Let $P(\bar{x}, \bar{y})$ denote the center of mass of D.

Let V be the solid region generated by rotating D about the y-axis.

Let C denote the circle which is the path of P as D rotates about the y-axis.

Prove the Theorem of Pappus:

$$\text{Volume}(V) = \text{Length}(C) \times \text{Area}(D).$$

Question 9. [10 marks]

Let

$$\mathbf{F}(x, y, z) = \langle 3x^2y + 3\sin(yz), x^3 + 3xz\cos(yz), 3xy\cos(yz)\rangle$$

be a vector field on $\mathbb{R}^3$.

(i) Determine if $\mathbf{F}$ is a conservative field.
If it is a conservative field, find a potential function $f(x, y, z)$, i.e. $\nabla f = \mathbf{F}$.
If it is not a conservative field, give a proof.

(ii) Let C denote the directed segment $\mathbf{r}(t) = \langle t(t+1), t^3, (1-t^2)\sin(\pi t)\rangle$ for $0 \le t \le 2$.
Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Question 10. [10 marks]

Let

$$P = \frac{x-y}{x^2+4y^2} \quad \text{and} \quad Q = \frac{x+4y}{x^2+4y^2}.$$

(i) Compute $\oint_{C_1} P dx + Q dy$ where C_1 is the directed closed path which is an ellipse given by $\mathbf{r}(t) = \langle 2\cos t, \sin t\rangle$ for $0 \le t \le 2\pi$.

(ii) Compute $\oint_{C_2} P dx + Q dy$ where C_2 is the directed circle $\mathbf{r}(t) = \langle 30\cos t, 30\sin t\rangle$ for $0 \le t \le 2\pi$.
(Hint: Do not compute directly.)

Question 11. [10 marks]

(i) Let $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3\rangle$ be a vector field.
Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the unit sphere

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

(ii) Let $\mathbf{G}(x, y, z) = \langle x^2, y^2, z^2\rangle$ be a vector field.
Compute $\iint_{S_1} \mathbf{G} \cdot d\mathbf{S}$ where S_1 is the ellipsoid

$$S_1 = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{(x-5)^2}{2^2} + \frac{(y-6)^2}{3^2} + (z+8)^2 = 1\right\}.$$

You may assume that the volume of the ellipsoid is 8π.

END OF PAPER