Student Number:	A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2012-2013

MA1101R LINEAR ALGEBRA I

April/May 2013 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **FOUR (4)** questions and comprises **NINETEEN (19)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
Total	

Question 1 [25 marks]

Let $S = \{(1,0,1,0), \quad (0,1,0,1), \quad (1,1,0,0)\}$.

(i) (4 marks) Show that the set S is linearly independent.

(ii) (3 marks) What is the dimension of the vector space $V = \text{span}(S)$? Give a brief explanation.

(iii) (4 marks) Express the vector $\boldsymbol{v} = (7,-1,3,-5)$ as a linear combination of the vectors in S and write down the coordinate vector $(\boldsymbol{v})_S$.

(iv) (3 marks) Find the vector $\boldsymbol{w} \in \mathbb{R}^4$ such that the coordinate vector $(\boldsymbol{w})_S = (2,3,-6)$.

(v) (3 marks) Suppose T is another basis for V such that the transition matrix from S to T is given by $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find the coordinate vector $(\boldsymbol{w})_T$ relative to T for the vector $\boldsymbol{w}$ in part (iv).

(vi) (4 marks) Determine the basis T in part (v), i.e. find all the vectors in T.

(vii) (4 marks) Is it possible to find a subspace U of $\mathbb{R}^4$ such that

$$V \subseteq U \subseteq \mathbb{R}^4 \text{ but } V \neq U \text{ and } U \neq \mathbb{R}^4?$$

Justify your answer.

Use the space below to write your answer and working

(More working spaces for Question 1)

(More working spaces for Question 1)

(More working spaces for Question 1)

Question 2 [25 marks]

Let $\boldsymbol{A} = (\boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3 \ \ \boldsymbol{a}_4 \ \ \boldsymbol{a}_5)$ be a 4×5 matrix where $\boldsymbol{a}_i$ denotes the ith column of $\boldsymbol{A}$. Suppose the reduced row echelon form of $\boldsymbol{A}$ is given by

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) (7 marks) Write down a basis for each of the row space, column space and nullspace of the matrix $\boldsymbol{A}$.

(ii) (2 marks) Write down two vectors to extend the basis for the row space of $\boldsymbol{A}$ in part (i) to a basis for $\mathbb{R}^5$.

(iii) (3 marks) Find a 5×5 matrix without zero rows or repeating rows that has the same row space as $\boldsymbol{A}$.

(iv) (3 marks) Is $\{\boldsymbol{a}_3, \ \boldsymbol{a}_4, \ \boldsymbol{a}_5\}$ a basis for the column space of $\boldsymbol{A}$? Justify your answer.

(v) (4 marks) By pre-multiplying $\boldsymbol{A}$ with an invertible 4×4 matrix $\boldsymbol{B}$, is it necessary that $\boldsymbol{BA}$ has the same row space as $\boldsymbol{A}$? Justify your answer.

(vi) (3 marks) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be a linear transformation with $\boldsymbol{A}$ above as the standard matrix. Suppose we are given

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find $T\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}\right)$.

(vii) (3 marks) For the linear transformation T in part (vi), is there enough information to determine its formula? Justify your answer.

Use the next three pages to write your answer and working

(Working spaces for Question 2)

(More working spaces for Question 2)

(More working spaces for Question 2)

Question 3 (a) [13 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

(i) (4 marks) Find the characteristic polynomial of $\boldsymbol{A}$. Hence, or otherwise, show that the eigenvalues of $\boldsymbol{A}$ are -2 and 4.

(ii) (4 marks) Find a basis for each of the eigenspaces of $\boldsymbol{A}$.

(iii) (2 marks) Is $\boldsymbol{A}$ diagonalizable? Justify your answer.

(iv) (3 marks) Find a square matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^3 = \boldsymbol{A}$. (You may leave your answer as a product of matrices.)

Use the space below to write your answer and working

(More working spaces for Question 3a)

Question 3 (b) [6 marks]

Find the least squares solution of the linear system

$$\begin{cases} x \qquad + 2z = 1 \\ \qquad y + 3z = 0 \\ -x + y + z = 0 \\ \quad - y - 3z = 1 \ . \end{cases}$$

Use the space below to write your answer and working

Question 3 (c) [6 marks]

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size. Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{AB}$ associated with eigenvalue λ.

(i) If $\lambda \neq 0$, show that $\boldsymbol{Bx}$ is an eigenvector of $\boldsymbol{BA}$ with eigenvalue λ.

(ii) If $\lambda = 0$, is $\boldsymbol{Bx}$ an eigenvector of $\boldsymbol{BA}$ with eigenvalue λ? Justify your answer.

Use the space below to write your answer and working

Question 4 (a) [13 marks]

Let $V = \{(w, x, y, z) \mid w - x + y - z = 0\}$.

(i) (3 marks) Write down the vector space V explicitly. Hence, find a basis for V.

(ii) (6 marks) Use the Gram-Schmidt process to find an orthogonal basis for V.

(iii) (2 marks) Extend the set obtained in (ii) to an orthogonal basis for $\mathbb{R}^4$.

(iv) (2 marks) Find the projection of $(2, -2, 2, -2)$ onto V.

Use the space below to write your answer and working

(More working spaces for Question 4a)

Question 4 (b) [6 marks]

Let $\boldsymbol{A}$ be a square matrix of order n such that for any $\boldsymbol{u} \in \mathbb{R}^n$,

$$\|\boldsymbol{A}\boldsymbol{u}\| = \|\boldsymbol{u}\|.$$

(i) Prove that $\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$ for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

(ii) Using (i) or otherwise, prove that $\boldsymbol{A}$ is an orthogonal matrix.

Use the space below to write your answer and working

Question 4 (c) [6 marks]

Let $\boldsymbol{A}$ be a square matrix of order n such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(i) Prove that $\boldsymbol{A}$ is diagonalizable.

(Hint: First show that the only possible eigenvalues of $\boldsymbol{A}$ are 0 and 1.)

(ii) Prove that $\text{rank}(\boldsymbol{A}) = \text{tr}(\boldsymbol{A})$.
(Here $\text{tr}(\boldsymbol{A})$ denotes the trace of $\boldsymbol{A}$, which is given by the sum of the diagonal entries of $\boldsymbol{A}$.)

Use the space below to write your answer and working

(More working spaces. Please indicate the question numbers clearly.)

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[END OF PAPER]