NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER EXAMINATION FOR THE DEGREE OF B.SC.

SEMESTER 1 EXAMINATION 2012–2013

MA2213 Numerical Analysis I

December 2012– Time allowed : 2 hours

Instructions to Candidates

1. This examination paper contains a total of **Five (5)** questions and comprises **Five (5)** printed pages.
2. Answer **ALL** questions.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
4. All questions carry equal marks.

Question 1 [20 marks]

(a) The numbers a, b and θ have been rounded to

$$\bar{a} = 3.12, \quad \bar{b} = 5.14, \quad \bar{\theta} = 0.973.$$

(i) Give the bounds for the absolute and relative errors in $\bar{a}$, $\bar{b}$ and $\bar{\theta}$.

(ii) Estimate the interval in which the exact value

$$A = \frac{1}{2}\, a\, b \sin(\theta)$$

can be found.

(b) (i) Let

$$y_k = 2^k \sin(2^{-k}\pi), \quad k = 1, 2, \cdots .$$

By writing $\alpha = 2^{-k}\pi$ and using the trigonometric identity

$$\sin(\frac{\alpha}{2}) = [\frac{1}{2}(1 - \cos(\alpha))]^{1/2},$$

show that

$$y_{k+1} = 2^{k+1}\left(\sqrt{\frac{1}{2}\left(1 - \sqrt{1 - (2^{-k} y_k)^2}\right)}\right). \tag{1}$$

(ii) Since

$$\lim_{n \to \infty}\left(n \sin(\frac{\pi}{n})\right) = \pi,$$

with an appropriate y_1, the value of π may be obtained as the limit of the iterative scheme in (1). Explain why in actual computation this iterative scheme fails to converge to π.

(iii) Suggest a modification to (1) so that y_k converges to π as k increases.

Question 2 [20 marks]

(a) Use Gaussian elimination with partial pivoting and three-digit rounding arithmetic to solve the following linear system:

$$\begin{aligned} 3.03x_1 - 12.1x_2 + 14x_3 &= -119, \\ 2.11x_1 - 13.2x_2 + 21x_3 &= -139, \\ -3.03x_1 + 12.1x_2 - 7x_3 &= 120. \end{aligned}$$

(b) Let $p(x)$ be a polynomial of degree at most n that interpolates $f(x)$ at distinct points x_0, x_1, $\cdots$, x_n in the interval $[0,\ 1]$. Suppose we want to use $\int_0^1 p(x)dx$ as an estimate of $\int_0^1 f(x)dx$. Assume that $|f^{(n+1)}(x)| < M$ on $[0,\ 1]$. What upper bound can be given for the error

$$\left| \int_0^1 f(x)dx - \int_0^1 p(x)dx \right|$$

if nothing is known about the location of the points x_0, x_1, $\cdots$, x_n.

Question 3 [20 marks]

(a) Let $f(x)$ be continuous on $[a,\ b]$, and x_0, x_1, $\cdots$, x_n be $n+1$ distinct points in $[a,\ b]$. Assume that $p(x)$ is the polynomial of degree at most n interpolating $f(x)$ at points x_0, x_1, $\cdots$, x_n. For any $x \in \mathbf{R}$, compute the values of

$$\sum_{i=0}^{n} l_i(x) \quad \text{and} \quad (f(x) - p(x)) - \sum_{i=0}^{n} [f(x) - f(x_i)] l_i(x),$$

where

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \quad i = 0, 1, \cdots, n.$$

(b) Let $p(x)$ and $q(x)$ be polynomials with minimal degrees interpolating the following data, respectively,

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

and

x	-2	0	1	3	11
$f(x)$	1	11	16	-4	2999

Compute

$$\frac{q(x) - p(x)}{(x+2)x(x-1)(x-3)} - f[1, 0, 3, 11, -2],$$

where $f[1, 0, 3, 11, -2]$ is the 4^{th} order divided difference of $f(x)$ at the points $1, 0, 3, 11, -2$.

Question 4 [20 marks]

(i) Let

$$h = \frac{b-a}{2n},$$

and

$$x_i = a + ih, \quad i = 0, 1, 2, \cdots, 2n.$$

Given

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = \frac{h}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})] - \frac{h^5}{90} f^{(4)}(\xi_i),$$

where

$$\xi_i \in (x_{i-1},\ x_{i+1}),$$

show that

$$\int_a^b f(x)dx = I_n(f) - \frac{h^4}{180}(b-a)f^{(4)}(\xi),$$

where

$$I_n(f) = \frac{h}{3}[f(a) + 4\sum_{i=0}^{n-1} f(x_{2i+1}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + f(b)],$$

and $\xi \in (a, b)$.

(ii) Suppose

$$f(x) = \begin{cases} \dfrac{8}{\pi^3}x^3 + \dfrac{4}{\pi}x - 3, & 0 \le x \le \dfrac{\pi}{2}, \\ \cos(x), & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$$

Find the smallest value of n such that

$$\left| \int_0^\pi f(x)dx - I_n(f) \right| < 10^{-5}.$$

Question 5 [20 marks]

(a) Let $T(f,h)$ denote the approximation of

$$I(f) := \int_a^b f(x)dx$$

using trapezoidal rule for step size $h = \dfrac{b-a}{2^n}$. Suppose that the corresponding error in the integration formula has the asymptotic expansion

$$I(f) - T(f,h) = \alpha_1 h^{p_1} + \alpha_2 h^{p_2} + \alpha_3 h^{p_3} + \cdots ,$$

where $0 < p_1 < p_2 < p_3 < \cdots$, and the coefficients $\alpha_1 \neq 0$, α_2, α_3, $\cdots$, are all independent of h.

(i) Derive a better approximation to $I(f)$ using $T(f,h)$ and $T(f, \frac{h}{2})$.

(ii) The following results were obtained using composite trapezoidal rule to estimate the value of

$$I(f) = \int_0^\pi \sin(x)dx$$

with 2, 4 and 8 subintervals:

n	$T(f,h)$
2	1.57079633
4	1.89611890
8	1.97423160

Using the result in (i) with $p_1 = 2$, calculate a better approximation for $I(f)$.

(b) Let $f(x) = x^3 - 2x - 5$. It is known that the function $f(x)$ has a unique zero α in the interval $(2,3)$. Discuss the convergence of the following iterative schemes for finding the zero α of $f(x)$:

(i) $x_{n+1} = \dfrac{x_n^3 - 5}{2}$.

(ii) $x_{n+1} = (2x_n + 5)^{1/3}$.