NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2012-2013

MA2108
MATHEMATICAL ANALYSIS I

November 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **FOUR (4)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions.

3. This is a closed book examination. A help sheet up to A4 size is allowable.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) For each of the following sequences, either find the limit or show that the limit does not exist.

(i) $\left(\dfrac{2n+1-n^2}{n^2-2n+6}\right)$.

(ii) $\left(\left(1+\dfrac{1}{n+2}\right)^{2n}\right)$.

(iii) $\left(\sqrt{(n+a)(n+b)}-n\right)$, where $a>0$ and $b>0$.

(b) Alternate the terms of the sequences $(1+1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2,-1,3/2,-1/2,4/3,-1/3,5/4,-1/4,\ldots).$$

Determine the values of $\limsup(x_n)$ and $\liminf(x_n)$.

Question 2 [30 marks]

(a) Determine the convergence or divergence of each of the following series. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+1}{3n^3+2n}$.

(ii) $\displaystyle\sum_{n=1}^{\infty}4^n\left(\frac{n}{n+2}\right)^{n^2}$.

(iii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(n!)(n+1)!}{(2n)!}$.

(b) Consider the series

$$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\frac{1}{3}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}+\frac{1}{5}-\frac{1}{14}-\frac{1}{16}-\frac{1}{18}+---\cdots,$$

where the signs are given as $+---$. Let a_n be its n-th term and let $S_n=\sum_{k=1}^{n}a_k$ be the partial sum.

(i) Determine the n-term a_n.

(ii) Use (i) or other methods, show that the sequence (S_{4n}) converges.

(iii) Does the series converge? Justify your answer.

Question 3 [30 marks]

(a) Evaluate the following limits or show that they do not exist.

(i) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \quad (x>0)$.

(ii) $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} \quad (x>0)$.

(iii) $\lim_{x\to 2^+} \frac{[x]-x}{x-2}$.

(Here $[x]$ is the greatest integer less than or equal to x.)

(b) Let f be a continuous function on $[0,\infty)$ such that it is uniformly continuous on $[a,\infty)$ for some positive number a. Prove that f is uniformly continuous on $[0,\infty)$.

(c) Given any real number c, is it possible to rearrange the terms of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ so that the new sum is exactly c?

(i) Write your answer to this question by Yes/No.

(ii) Prove your claim in (i).

(iii) Can you give two rearrangements $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ such that $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n = c$ but $\sum_{n=1}^{\infty} x_n$ **cannot** be obtained by rearranging **finite terms** from $\sum_{n=1}^{\infty} y_n$? Justify your answer.

Question 4 [20 marks]

(a) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function that does not take on any of its values twice and with $f(0) > f(1)$. Show that f is strictly decreasing on $[0,1]$.

(b) Let $f\colon [a,b] \to \mathbb{R}$ be a (not necessarily continuous) function with the property that for every $x \in [a,b]$, there exists a positive number δ_x such that the function f is bounded on $(x-\delta_x, x+\delta_x) \cap [a,b]$. Prove that f is bounded on $[a,b]$.

(c) Let $\sum_{n=1}^{\infty} a(n)$ be a series such that $(a(n))$ is a decreasing sequence of positive numbers. Prove that $\sum_{n=1}^{\infty} a(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges.

END OF PAPER