NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION (2012–2013)

MA1521 Calculus for Computing

December 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

Let $f(x, y) = x^2 + 2y^2 - x^2y^2$.

(i) Find the coordinates of all the critical points of f.

(ii) Determine whether the critical points of f are local maximums, local minimums or saddle points. Justify your answers.

Question 2 [10 marks]

Using Lagrange multipliers method or otherwise, prove that among all triangles with a fixed perimeter p, the equilateral one has the largest area.

(You may use Heron's formula for the area:

$$A = \sqrt{s(s-x)(s-y)(s-z)},$$

where $s = p/2$ and x, y, z are the lengths of the sides.)

Question 3 [14 marks]

Find the following definite integrals.

(a) $\displaystyle\int_0^1 x^2 \left(\sqrt[3]{1-x}\right)\, dx.$

(b) $\displaystyle\int_2^\infty \frac{1}{x^4 - 1}\, dx.$

Question 4 [12 marks]

Determine whether each of the series is convergent or divergent. Justify your answers.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{(\ln n)^n}.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+1) - \ln n}{n}.$

Question 5 [14 marks]

(a) Let $f(x) = \dfrac{1}{x^2 - 3x - 4}$.

(i) Find the Taylor series of $f(x)$ about $x = 1$.

(ii) Using (i) or otherwise, evaluate $f^{(1521)}(1)$.

(b) Let $I(m, n) = \displaystyle\int_0^1 x^m (\ln x)^n \, dx$, where m and n are nonnegative integers.

(i) Prove that $I(m, n+1) = -\dfrac{n+1}{m+1} I(m, n)$.

(ii) Using (i) or otherwise, evaluate $\displaystyle\int_0^1 (x \ln x)^{1521} \, dx$.

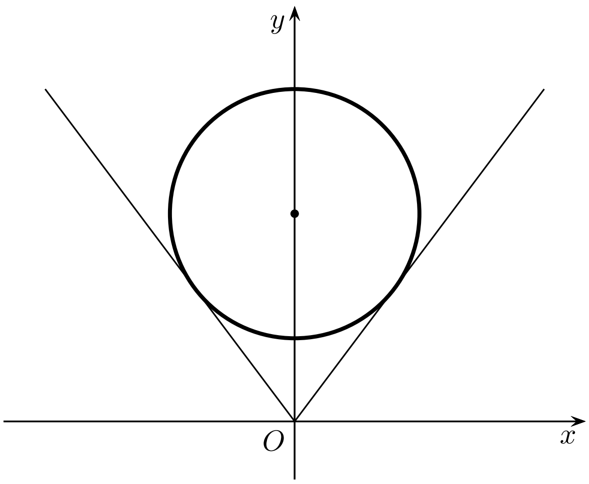
Question 6 [16 marks]

The circle with radius 3 shown in the figure touches the curve $y = \dfrac{4}{3}|x|$.

(i) Find the equation of the circle.

(ii) Find the volume of the solid formed by rotating the region enclosed by the circle about the x-axis.

(iii) Find the area of the surface formed by rotating the circle about the x-axis.



Question 7 [16 marks]

Solve the following differential equations.

(a) $(3x^2 + 2xy - y^2) + (x^2 - 2xy)\,\dfrac{dy}{dx} = 0, \quad y = 2$ at $x = 1$.

(b) $\dfrac{d^2y}{dx^2} - 2\,\dfrac{dy}{dx} + 2y = e^x \sin x, \quad y = 1$ and $\dfrac{dy}{dx} = 1$ at $x = 0$.

Question 8 [8 marks]

Suppose the function $F(x, y)$ is **homogeneous of degree $\boldsymbol{n}$**; that is,

$$F(tx, ty) = t^n F(x, y) \qquad \text{for all } t \in \mathbb{R} \setminus \{0\}.$$

Assume that all the first and second order partial derivatives of $F(x, y)$ are continuous.

Prove that

(i) $x\,\dfrac{\partial F}{\partial x} + y\,\dfrac{\partial F}{\partial y} = nF(x, y)$.

(ii) $x^2\,\dfrac{\partial^2 F}{\partial x^2} + 2xy\,\dfrac{\partial^2 F}{\partial x\,\partial y} + y^2\,\dfrac{\partial^2 F}{\partial y^2} = n(n-1)F(x, y)$.

End of Paper