NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2012-2013

MA1507 Advanced Calculus

December 2012 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is an open book examination. Candidates are allowed to refer to any written/printed material.

2. This examination paper consists of **SEVEN (7)** questions and comprises **THREE (3)** printed pages.

3. Answer **ALL** questions. This exam carries a total of **60** marks.

4. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Section **A** *contains* **FOUR (4)** *questions and carries a total of* 40 *marks. Answer* **ALL** *questions.*

Question 1.

(a) Explain why the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+5y^4}.$$

(b) Find the tangent plane of the surface

$$x^2 - 2xz + z^2 + y^2 z = 4$$

at the point $(1, 2, 1)$.

Question 2.

(a) Suppose $z = f(u, v)$, where $f(u, v)$, $f_u(u, v)$ and $f_v(u, v)$ are differentiable functions of u and v. Suppose $u = x^2 y$ and $v = y^2$. Using Chain Rule, compute the following partial derivatives. Express your answers in terms of x, y and the partial derivatives (including those of higher order) of f with respect to u and v.

(i) $\dfrac{\partial z}{\partial x}$

(ii) $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$

(b) An insect is moving clockwise around the circle $x^2 + y^2 = 4$ on the xy-plane. The temperature function $T(x, y)$ (experienced by the insect) is given by

$$T(x, y) = 2xy + x^2 - y^2, \quad (x, y) \in \mathbb{R}^2$$

where T is measured in degree Celsius and x, y are measured in meters. What is the rate of change of the temperature (in degree Celsius per meter) when the insect is at the point $(\sqrt{3}, 1)$.

Question 3.

(a) By changing the order of integration, evaluate the following iterated integral:

$$\int_0^1 \int_{x^2}^1 2x \cos(y^2)\, dy\, dx.$$

(b) Let E be the solid bounded below by the paraboloid $z = \sqrt{2}(x^2 + y^2)$ and above by the upper hemisphere $z = \sqrt{3 - x^2 - y^2}$. Find the volume of E.

Question 4.

Using the method of Lagrange multipliers, find the point(s) on the surface $z = y^2 - x^2$ that is closest to the point $(0, 0, 1)$.

Section **B** *contains* **THREE (3)** *questions and carries a total of* 20 *marks. Answer* **ALL** *questions.*

Question 5.

Consider the force field $\mathbf{F}(x, y, z) = \langle 3x^2, 2xz - y, z\rangle$.

(i) Is $\mathbf{F}$ a conservative vector field? Justify your answer.

(ii) Find the work done by $\mathbf{F}$ in moving a particle along the straight line from $(1, 1, 1)$ to $(1, 2, 5)$.

(iii) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the intersection of the plane $x + z = 1$ and the cylinder $x^2 + y^2 = 1$ with anti-clockwise orientation when viewed from the positive z-axis.

Question 6.

Consider the vector field

$$\mathbf{F}(x, y, z) = \left\langle x^2 + y^2 + z^2, \quad -2xy + \sin z, \quad 2z \right\rangle, \ (x, y, z) \in \mathbb{R}^3.$$

Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the following oriented surface S:

(a) S is the disk $x^2 + y^2 \leq 1$ on the xy-plane with unit normal $\mathbf{n} = \langle 0, 0, -1\rangle$.

(b) S is the disk $y^2 + z^2 \leq 1$ on the yz-plane with unit normal $\mathbf{n} = \langle -1, 0, 0\rangle$.

(c) S is the sphere $x^2 + y^2 + (z - 5)^2 = 1$ with positive orientation (i.e. outward pointing normal).

Question 7. Let

$$\mathbf{F}(x, y, z) = \left\langle e^{\sin z} - 7, \quad 0, \quad -3xy^2 \right\rangle, \ (x, y, z) \in \mathbb{R}^3.$$

Compute the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface $z = 1 - x^2 - y^2$ with $z \geq 0$, oriented with downward pointing normal.

END OF PAPER