NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2012-2013

MA1506 Mathematics II

November/December 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in the examination paper. Marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
4. Two A4 handwritten double-sided helpsheets are allowed.

Answer all the questions.

Marks for each question are indicated at the beginning of the question.

Question 1 [20 marks]

(a) By using the substitution $u = e^{2y}$, solve the differential equation

$$2xe^{2y}\frac{dy}{dx} = 3x^4 + e^{2y}, \qquad x > 0.$$

(b) Solve the differential equation

$$\frac{d^2y}{dx^2} = 16\text{sin}x\text{cos}x, \quad y'(0) = 0,\ y(0) = 0.$$

(c) In a piece of wood, it was found that 85% of the Carbon-14 had decayed. Using the fact that Carbon-14 has a half-life of 5730 years, (the time required for a given amount of Carbob-14 to decay to 50% of its original amount). Find the age (in years) of the wood.

(d) The growth $P(t)$ of a certain organization at time t can be modeled by the differential equation

$$\frac{dP}{dt} = P(a - b\ell nP),$$

where a and b are positive constants. Suppose $P(0) = P_0 > 0$, find the limit (in terms of a and b) of $P(t)$ as $t \to \infty$.

Question 2 [20 marks]

(a) The motion of a particle is given by the differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + 2x = 0, \qquad b > 0.$$

Find the range of values of b for which the motion is over-damped.

(b) (i) Let $x(t)$ be a differentiable function of t and let $y = \frac{dx}{dt}$. Use the chain rule of differentiation to prove that

$$\frac{d}{dx}(y^2) = 2\frac{dy}{dt}.$$

(ii) Let $x(t)$ be the solution to the differential equation

$$\ddot{x} + x = 0, \quad x(0) = 10m, \ \dot{x}(0) = 5m/s,$$

where $\dot{x}(t) = \frac{dx}{dt}$. Find the value of

$$(\dot{x}(t))^2 + (x(t))^2$$

for all $t > 0$.

(c) Let f be a differentiable function such that $u(x,t) = f(x + at)$ is a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

Find the possible values of a.

(d) (i) Prove that $u(x,t) = e^{-\pi^2 t}\sin\pi x$ is a solution to the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ 0 < t < \infty;$$
$$u(0,t) = u(1,t) = 0, \quad 0 \le t < \infty;$$
$$u(x,0) = \sin\pi x, \quad 0 \le x \le 1.$$

(ii) Use the information in (i) to solve the following partial differential equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \ 0 < t < \infty;$$
$$u(0,t) = u(1,t) = 0, \quad 0 \le t < \infty;$$
$$u(x,0) = 5\sin\pi x, \quad 0 \le x \le 1.$$

Question 3 [20 marks]

(a) Let A be a non-singular matrix. Suppose λ is a non-zero eigenvalue of A, prove, or disprove (by giving an example), that $\frac{1}{\lambda}$ is an eigenvalue of the inverse A^{-1} of A.

(b) Let

$$M = \begin{bmatrix} -\frac{1}{2} & a \\ b & c \end{bmatrix}$$

be a matrix representing an anti-clockwise rotation about the origin through an angle θ, where $0 < \theta < \pi$.

(i) Find the values of a, b and c.

(ii) Find M^{60} in its simplest matrix form.

(c) (i) Let

$$A = \begin{bmatrix} a & b \\ 1 & -2 \end{bmatrix}.$$

Find the values of a and b if A has eigenvectors

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(ii) With the values of a and b found in (i) above, find A^{2n} for $n = 1, 2, 3, \cdots$, express your answer in its simplest matrix form.

(d) (i) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\mathbf{i} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$T\mathbf{j} = 2\mathbf{i} - \mathbf{k}$$

$$T\mathbf{k} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the x, y, z-axis, respectively.

Determine whether T is non-singular. Justify your answer.

(ii) Write a MATLAB programme to solve the following system of equations

$$x + y + z = 0$$

$$x - z = 1$$

$$x - y + z = 6.$$

Question 4 [20 marks]

(a) Solve the system of differential equations

$$\frac{dx}{dt} = -x - 4y;$$

$$\frac{dy}{dt} = -3x - 2y.$$

Sketch the phase plane diagram for the system and classify the equilibrium point of the system as one of the six types discussed in class.

(b) Solve the differential equation

$$\frac{dx}{dt} + x = f(t), \quad x(0) = 0,$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 \leq t < 2 \\ 0 & \text{if } 2 < t < \infty. \end{cases}$$

(c) The Laplace transform $I(s)$ of the current $i(t)$ in a certain circuit is given by

$$I(s) = \frac{1+s}{s(s^2+s+1)}.$$

(i) Find $i(t)$ for $t > 0$.

(ii) Determine $\lim_{t\to\infty} i(t)$.

END OF PAPER