NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2012-2013

MA1104 Multivariable Calculus

December 2012 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring **TWO (2)** pieces of A4-sized two-sided help sheets into the examination room.

2. This examination paper consists of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions. This exam carries a total of **70** marks.

4. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Section **A** *consists of* **FOUR (4)** *questions and carries a total of* 40 *marks. Answer* **ALL** *questions.*

Question 1.

(a) Find parametric equations of the tangent line to the curve that is the intersection of $x^2 - 2y + z^2y + xz = 1$ and $xy + 3y^2z = 4$ at the point $(1, 1, 1)$.

(b) Suppose the temperature at a point (x, y, z) is given by $T(x, y, z) = e^{-x^2} + xyz$, where T is measured in degree Celcius and x, y, z in meters.

(i) Find the rate of change of temperature at the point $(0, 2, 1)$ in the direction toward the point $(1, 1, 1)$.

(ii) In which direction does the temperature increase fastest at $(0, 2, 1)$?

(iii) Find the maximum rate of increase at $(0, 2, 1)$.

Question 2. Let $z = f(u, v)$ be a differentiable function with continuous second order partial derivatives. Suppose $u = xy + y$ and $v = x^2 + y$. Express the following partial derivatives in terms of x, y and the partial derivatives of f (including those of of higher order).

(i) $\dfrac{\partial z}{\partial x}$

(ii) $\dfrac{\partial^2 z}{\partial x^2}$

Question 3.

(a) By changing the order of integration, evaluate the following double integral:

$$\int_0^1 \int_{1-x}^{\sqrt{1-x}} e^{y^2/2 - y^3/3} \; dy \, dx.$$

(b) Find the volume of the solid below the surface $\sqrt{3}\, z = \sqrt{x^2 + y^2}$ and above the surface $x^2 + y^2 + z^2 = 2z$.

Question 4.

(a) Identify all the points at which the function

$$f(x, y, z) = xyz + \frac{z}{12}$$

is maximum and minimum respectively, subject to the conditions that $x+y+z = 1$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.

(b) Let D be the region on the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 2\sqrt{x}$, $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Evaluate the double integral

$$\iint_D \frac{2x^2 + y^2}{xy} \, dA.$$

Section **B** *consists of* **THREE (3)** *questions and carries a total of* 30 *marks. Answer* **ALL** *questions.*

Question 5.

Let $\mathbf{F}(x, y) = \langle 2x^5 - 3x^2y^2, -2x^3y + Ax \rangle$ for some constant $A \in \mathbb{R}$. Let C be the curve given by $\mathbf{r}(t) = \langle t, \sin t \rangle$, $0 \leq t \leq \pi$.

(i) Suppose $A = 0$. Determine whether $\mathbf{F}$ is conservative. Hence or otherwise, evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

(ii) Suppose $A = \pi$. Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

Question 6. Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a vector field defined on $\mathbb{R}^2$ such that $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives. Let C be the circle $x^2 + y^2 = a^2$ for some positive constant a, and $\mathbf{N}(x, y) = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$. Show that

$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \iint_D (P_x + Q_y) \, dA$$

where D is the region on the xy-plane bounded by C.

Question 7.

Let S be the closed surface in the xyz-space defined via the cylindrical coordinate (r, θ, z) as follows:

$$\begin{aligned} r &= 2 + \cos u \\ \theta &= t \\ z &= \sin u \end{aligned}$$

where $0 \leq t \leq 2\pi$, $0 \leq u \leq 2\pi$.

Compute the volume of the solid bounded by the surface S.

END OF PAPER