NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION (2012–2013)

MA1102R Calculus

November 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **TEN (10)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [11 marks]

(a) Find the limit $\lim_{x \to 1} \left(\dfrac{5}{1-x^5} - \dfrac{4}{1-x^4} \right)$.

(b) Use the ϵ, δ-definition to prove that $\lim_{x \to -1} (x^3 + x^2 + x) = -1$.

Question 2 [8 marks]

Consider the function $f(x) = (x^2 - 4x + 5)e^x$ on $\mathbb{R}$.

(i) Find the open intervals on which f is increasing and decreasing.

(ii) Find the coordinates of the local maximum and local minimum points of f, if any.

(iii) Find the open intervals on which f is concave up and concave down.

(iv) Find the coordinates of the inflection points of f, if any.

Question 3 [11 marks]

(a) Define the functions

$$p(x) = (x-1)(x-3)(x-5) \quad \text{and} \quad q(x) = (x-2)(x-4)(x-6).$$

Prove that the equation $p(x) + q(x) = 0$ has at least three real roots.

(b) Consider the curve

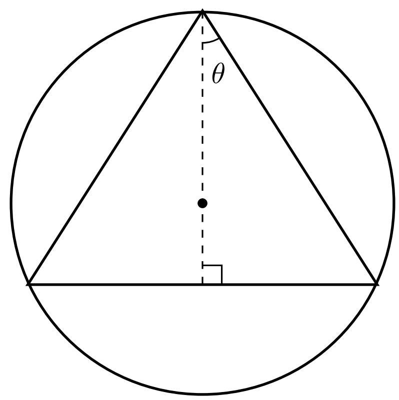
$$y = \frac{(4-x^2)^x}{x^{4-x^2}}.$$

Find the equation of its tangent line at the point $(1, 3)$.

Question 4 [8 marks]

(i) An isosceles triangle is inscribed in a circle of radius 1. Let the angle at the apex of the isosceles triangle be 2θ. Show that the area of the triangle is given by $4\sin\theta\cos^3\theta$.

(ii) Using the result in part (i), prove that among all isosceles triangles inscribed in a circle of radius 1, the equilateral one has the largest area.



Question 5 [12 marks]

Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}\,(1+x)}\,dx.$

(b) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$

Question 6 [14 marks]

(a) Find the nonzero real constant a for which $\displaystyle\lim_{x\to 0}(\cos x)^{\frac{2a}{x^2}} = 2\int_a^{\infty} xe^{-x}\,dx.$

(b) Let f be a function that is continuous on $\mathbb{R}$. Evaluate $\displaystyle\frac{d}{dx}\int_0^x tf(x^2-t^2)\,dt.$

Question 7 [12 marks]

(a) For $a > 0$, find the volume of the solid generated by revolving the region enclosed by the curve $y = \sqrt{x}$ and the line $y = ax$ about the x-axis. Hence find the value of a which gives a volume of 6π for the solid of revolution.

(b) Find the length of the curve $x = \frac{3}{4}\left(\frac{y^{4/3}}{2} - y^{2/3}\right)$ from $y = 1$ to $y = 27$.

Question 8 [8 marks]

(i) Solve the differential equation

$$\frac{dz}{dx} + \left(\frac{e^x}{1+e^x}\right) z = \frac{\sin x}{1+e^x}.$$

(ii) Using part (i), solve the following Bernoulli's equation with initial condition:

$$-3(1+e^x)\frac{dy}{dx} + e^x y = (\sin x)\, y^4, \qquad \text{where } y = 1 \text{ if } x = 0.$$

Question 9 [8 marks]

The downward velocity of a falling object is modeled by the differential equation

$$\frac{dv}{dt} = -0.002(v^2 - 4900),$$

where $v = v(t)$ denotes the velocity of the object at time t in meters per second. Initially the velocity of the object is $0\,\text{m/s}$.

(i) Derive an expression for the velocity v at time t.

(ii) Using part (i), find the terminal velocity that v eventually increases to.

Question 10 [8 marks]

For $a < b$, let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b) with $f(x)g'(x) \neq g(x)f'(x)$ for every $x \in (a, b)$. Suppose that $f(a) = f(b) = 0$ and f has no zeros in (a, b). Prove that if $g(a) \neq 0$ and $g(b) \neq 0$, then g has exactly one zero in (a, b).

End of Paper