NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2012-2013

MA1101R Linear Algebra I

November 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. Each question carries 25 marks.
3. Calculators may be used. However, you should lay out systematically the various steps in the calculations

Question 1

(a) [18 marks] Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices given below.

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -3 & 3 \\ 3 & -6 & -6 & 6 & 2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent. (**Note:** Show all your elementary row operations.)

(ii) Write down a basis for the row space of $\boldsymbol{A}$.

(iii) Are the column spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$ equal? Justify your answer.

(iv) Find a basis for and determine the dimension of the nullspace of $\boldsymbol{A}$.

(v) Without performing any computation, explain why we can conclude that $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.

(vi) Explain why $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b} \in \mathbb{R}^3$.

(b) [7 marks] Let S and T be two bases for $\mathbb{R}^3$ and $\boldsymbol{P}$ is the transition matrix from S to T. Suppose

$$\boldsymbol{P}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}, \quad \boldsymbol{P}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{P}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 7 \end{pmatrix}.$$

(i) Find the transition matrix from T to S.

(ii) If $\boldsymbol{w}$ is a vector in $\mathbb{R}^3$ such that $(\boldsymbol{w})_T = (1, 2, 1)$, find $(\boldsymbol{w})_S$.

Question 2

(a) [18 marks] Let

$$\boldsymbol{u_1} = (2, 0, -1), \quad \boldsymbol{u_2} = (3, 0, 1), \quad \boldsymbol{u_3} = (0, 1, 1), \quad \boldsymbol{b} = (1, 2, -1),$$

and $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

(i) Show that S is a basis for $\mathbb{R}^3$.

(ii) Find the coordinate vector of $\boldsymbol{b}$ relative to S.

(iii) If P is the plane in $\mathbb{R}^3$ span by $\boldsymbol{u_1}$ and $\boldsymbol{u_3}$, find the equation of P. (**Remark:** Do not use cross product of vectors.)

(iv) Find an orthogonal basis for $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$.

(v) Compute the distance from $\boldsymbol{b}$ to V.

(*Question 2 continues on next page...*)

(b) [7 marks] Let V and W be subspaces of $\mathbb{R}^n$. Recall the definition of $V+W$ as follows

$$V+W=\{\boldsymbol{v}+\boldsymbol{w} \mid \boldsymbol{v}\in V, \boldsymbol{w}\in W\}.$$

Define $V-W$ by

$$V-W=\{\boldsymbol{v}-\boldsymbol{w} \mid \boldsymbol{v}\in V, \boldsymbol{w}\in W\}.$$

(i) If $V=\text{span}\{(1,0,0)\}$ and $W=\text{span}\{(0,1,0),(0,0,1)\}$, show that every vector in $\mathbb{R}^3$ belongs to $V-W$.

(ii) Prove that in general, $V-W=V+W$.

Question 3

(a) [16 marks] Let $\boldsymbol{A}$ be the following matrix.

$$\boldsymbol{A}=\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}.$$

(i) Compute $\det(\boldsymbol{A})$ by finding a row echelon form of $\boldsymbol{A}$. (You will not be given any marks if you use any other method.)

(ii) Find 3 elementary matrices $\boldsymbol{E_1},\boldsymbol{E_2},\boldsymbol{E_3}$ such that $\boldsymbol{E_3E_2E_1A}$ is a matrix in row echelon form.

(iii) Find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T\boldsymbol{AP}=\boldsymbol{D}$ where $\boldsymbol{D}$ is a diagonal matrix. Write down $\boldsymbol{D}$ explicitly. (**Hint:** the eigenvalues of $\boldsymbol{A}$ are all integers.)

(b) [9 marks] Consider the following linear system

$$\begin{cases} x_1 + ax_2 + x_3 = 1\\ -x_1 + x_3 = 1\\ ax_1 + x_2 + ax_3 = 0\end{cases}$$

where a is a real number.

(i) If $a=-1$, show that the linear system is inconsistent. Find a least squares solution to the linear system in this case.

(ii) Determine all values of a such that the linear system has a unique solution.

(*Question 4 is on the next page...*)

Question 4

(a) [18 marks] Let

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}.$$

(i) If $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T_1\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_3} \quad \text{for all } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3.$$

Find the formula and the standard matrix for T_1.

(ii) If $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that

$$T_2(\boldsymbol{u_1}) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad T_2(\boldsymbol{u_2}) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad T_2(\boldsymbol{u_3}) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Find the formula and the standard matrix for T_2.

(iii) If $\boldsymbol{w}$ is a vector in $\mathbb{R}^3$ such that $(T_2 \circ T_1)(\boldsymbol{w}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find $\boldsymbol{w}$.

(iv) Find all $\boldsymbol{v}$ in $\mathbb{R}^3$ such that $(T_1 \circ T_2)(\boldsymbol{v}) = \mathbf{0}$.

(b) [7 marks]

(i) Let $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. Find the rank of $\boldsymbol{ab}^T$. (**Remark:** Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are matrices.)

(ii) Prove that an $m \times n$ matrix $\boldsymbol{A}$ has rank 1 if and only if $\boldsymbol{A} = \boldsymbol{ab}^T$ for some non zero matrices $\boldsymbol{a}$ and $\boldsymbol{b}$.

END OF PAPER