Ph.D. Qualifying Examination 2013 Jan (Analysis)

(1) A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if given any $x \in \mathbb{R}^n$, there exist $\delta, L > 0$ (depending on x) such that

$$|\phi(z) - \phi(y)| \le L|z - y| \text{ for all } z, y \in B_\delta(x) = \{t \in \mathbb{R}^n : |x - t| < \delta\}. \qquad (*)$$

If ϕ is locally Lipschitz on $\mathbb{R}^n$, show that for any compact set $K \subset \mathbb{R}^n$, there exists a constant $M > 0$ (depending on K) such that [10]

$$|\phi(x) - \phi(y)| \le M|x - y| \text{ for all } x, y \in K.$$

(2) Let $w \in L^1(\mathbb{R}^d)$ be strictly positive and $\{f_n\} : \mathbb{R}^d \to \mathbb{R}$ be (Lebesgue) measurable functions such that [12]

$$\lim_{m,n\to\infty} \int_{|f_n - f_m| > t} w(x)dx = 0 \quad \text{for any } t > 0.$$

Show that $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ that converges a.e. to a measurable function $g(x)$.

(3) Let Ω be an open connected subset of $\mathbb{R}^3$. Suppose $u \in C_0^2(\Omega)$ and $f \in C_0(\Omega)$ are such that

$$\Delta u - 2u = f \quad \text{on } \Omega \text{ where } \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Show that [5]

$$\int |f|^2 dx = \int |\Delta u|^2 + 4|\nabla u|^2 + 4|u|^2 dx \text{ where } \nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}).$$

(4) Compute (a) $\int_\gamma e^z/z^3 dz$, where $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(t) = e^{i6\pi t}$ and (b) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$, $a > 0$. [12]

(5) Let $f : \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \to \mathbb{R}$ be such that f is continuously differentiable with $\nabla f(0, 0) = (1, 2)$ and $f(0, 0) = 0$. Show that there exist $\varepsilon > 0$ and $\gamma \in C^1(-\varepsilon, \varepsilon)$ such that $\gamma(0) = 0$ and $f(x, \gamma(x)) = 0$ for $x \in (-\varepsilon, \varepsilon)$. Compute $\gamma'(0)$ if possible. [5]

(6) Explain in details with the help of the fact that $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$ why [10]

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\log 2.$$

(7) Let $w \in L^1[0,\pi]$ be nonnegative. Suppose $\{f_n\}$ is a sequence of (Lebesgue measurable) functions that converges a.e. to a function f. Suppose $\int_0^\pi |f_n|^2 w dx \to \int_0^\pi |f|^2 w dx < \infty$. Show that $\int_0^\pi |f_n - f|^2 w dx \to 0$. [10].

(8) Prove or disprove not more than six (6) of the following statements. [36]

(a) If ϕ is a function of bounded variation on $[a,b]$ for all $[a,b] \subset \mathbb{R}$ and g is a nondecreasing function on [0,1], then $\phi(g) \in BV[0,1]$.

(b) Let $\mathcal{C}$ be the Cantor set. Then $\chi_{\mathcal{C}}$ is Riemann integrable on $[0,1]$.

(c) If $\{f_j\}, f : (0,1) \to \mathbb{C}$ are integrable such that

$$\lim_{j\to\infty} \int_K f_j(x)dx = \int_K f(x)dx \text{ for all compact subset } K \text{ of } (0,1),$$

then

$$\lim_{j\to\infty} \int_{(0,1)} f_j(x)dx = \int_{(0,1)} f(x)dx.$$

(d) If $\{a_{i,j}\}_{i,j=1}^\infty$ is a collection of nonnegative real numbers, then

$$\sum_{i=1}^\infty \sum_{j=1}^\infty a_{i,j} = \lim_{N\to\infty} \sum_{\{i,j: i+j \le N\}} a_{i,j}.$$

(e) Let $\{f_n\} : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be analytic. If there exists f on $\mathbb{D}$ such that $f_n \to f$ uniformly on any compact subset of $\mathbb{D}$, then f is also analytic on $\mathbb{D}$.

(f) Let $u : G \to \mathbb{R}$ be a harmonic function where G is an open connected set in $\mathbb{C}$. Then there exists $v : G \to \mathbb{R}$ such that the function $f(z) = u(z) + iv(z)$ is analytic on G.

(g) If a function is piecewise differentiable on $[a,b]$, then it is of bounded variation on $[a,b]$.

(Note that a function f is said to be piecewise differentiable on $[a,b]$ if there exist a partition $a_0 = a < a_1 < \cdots < a_k = b$ of $[a,b]$ and $g_i : [a_{i-1}, a_i] \to \mathbb{R}$ such that its derivative g_i' is continuous on $[a_{i-1}, a_i]$ and $f = g_i$ on (a_{i-1}, a_i) for $i = 1, \cdots, k$.)

(h) Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable and for all $x \in \mathbb{R}^n$, define

$$h(x) = \limsup_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|^{1/2} dy.$$

Then $|f(x)| \le h(x)^2$ a.e..

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