Ph.D. Qualifying Examination 2012 August (Analysis)

(1) Let $n > 1$. For any $M > 1$, show that there exists $C_M > 1$ such that $\int_{MQ} |x|^{-1}dx \leq C_M \int_Q |x|^{-1}dx$ for any cube $Q \subset \mathbb{R}^n$ where MQ is the concentric cube with M times length as Q and $|x|$ is the Euclidean norm of x. [6]

(2) Let $< X, d >$ be a compact metric space. For any $0 < \alpha < 1$, let $C^\alpha(X)$ be the collection of continuous functions on X such that $\sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x,y)^\alpha} < \infty$. Suppose $\{f_n\}$ is a sequence in $C^\alpha(X)$ such that

$$\sup_n (\sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{d(x,y)^\alpha} + \sup_{x \in X} |f_n(x)|) < \infty.$$

Show that for each $0 < \beta < \alpha$, $\{f_n\}$ has a convergent subsequence converging uniformly to a function $f \in C^\beta(X)$. [5]

(3) Let $f(x) = \sqrt{x^2 + y^2 + z^2}$. Use the function ∇f on $D = \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 4\}$ to illustrate divergence theorem. (You will need to compute both integrals.) [7]

(4) Let $f : [0, 1] \to \mathbb{R}$ be measurable and $g \in L^1[0, 1]$ such that $\int_{|f(x)|>t} |g(x)|dx \leq 3/t^2$ for all $t > 0$, show that $\int_0^1 |f(x)|^p |g(x)|dx < \infty$ for $1 < p < 2$. [6]

(5) (i) Let $1 \leq p < \infty$ and w be a nonnegative integrable function on $[0, 1]$. Show that given any interval $I = [a, b] \subset [0, 1]$ and $\varepsilon > 0$, there exists a continuous function ϕ on $[0, 1]$ such that $\phi \geq \chi_I$ and $\int_0^1 |\phi|^p w(x)dx \leq \int_a^b w dx + \varepsilon$. Hence show that $C[0, 1]$ (space of continuous functions on $[0, 1]$) is dense in $L^p_w[0, 1]$ (with norm $(\int_0^1 |f|^p w dx)^{1/p}$). [11]

(ii) Assume that

$$\int_0^1 |f(x) - f_{av}|^p w(x)dx \leq C \int_0^1 |f'(x)|^p w(x)dx \quad \text{where } f_{av} = \int_0^1 f(x)dx \tag{1}$$

for all $f \in C^1[0, 1]$. Show that inequality (1) holds for all absolutely continuous functions f on $[0, 1]$ such that $f' \in L^p_w([0, 1])$. [10]

(6) Let $f, g \in L^p[a, b]$, $1 < p < \infty$. Show that the function $I(t) = \int_a^b |f(x) + tg(x)|^p dx$ is differentiable at $t = 0$ and compute its derivative. [10]

(7) Let Ω be an open connected subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be analytic. Is it true that f is conformal (on Ω)? Explain what "conformal" means and justify your answer. [5]

(8) For each $z \in \mathbb{C}$, evaluate [10]

$$\int_0^1 \int_0^{2\pi} \frac{1}{re^{i\theta} + z} d\theta dr.$$

(9) Prove or disprove **Six** (6) of the following statements. [30]

(a) If f is an entire function on $\mathbb{C}$, then the function $g(z) = \overline{f(\bar{z})}$ is also entire.

(b) Let Ω be an open connected set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2 : [0, 1] \to \Omega$ are piecewise differentiable such that for all $t \in [0, 1]$,

$|\gamma_1(t) - \gamma_2(t)| \leq \min\{t, 1 - t\}d(t)/2$ where $d(t) = \inf\{|\gamma_1(t) - z| : z \notin \Omega\}$,

then $\int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = \int_0^1 f(\gamma_2(t))\gamma_2'(t)dt$.

(c) If $\sum_{n=1}^{\infty} a_n(-1)^n$ converges, then $\sum_{n=1}^{\infty} a_n x^n$ converges to a C^∞ function on $(-1, 1)$.

(d) There exists a harmonic function f on $\{z = (x, y) : 0 < x^2 + y^2 < 1\}$ such that

(i) $\lim_{z \to z_0, |z|<1} f(z) = 1$ for all $|z_0| = 1$;

(ii) $\lim_{z \to 0} f(z) = -1$.

(e) If $f : \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable, then for each $x \in \mathbb{R}$, $f_x(y) = f(x, y)$ is Borel measurable on $\mathbb{R}$.

(f) The series $\sum_{k=1}^{\infty} \cos kx/k$ converges conditionally for almost all $x \in \mathbb{R}$.

(g) Let Ω be a connected open set in $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma : [0, 1] \to \Omega$ is a rectifiable curve, then the line integral $\int_\gamma f dz$ is defined.

(h) Let $\{a_n\}$ be a bounded sequence of positive real numbers such that $\lim_{n\to\infty} a_{n+1}/a_n = 1$, then $\{a_n\}$ is a convergent sequence.

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