NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2012-2013

Ph.D. QUALIFYING EXAMINATION

PAPER 1

ALGEBRA

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

(1) This examination paper contains **FIVE** questions and **THREE** printed pages.
(2) Each question has 4 parts, worth 5 points each, for a total of 20 points per question.
(3) Answer **ALL** questions.
(4) Please state any theorems used without proof clearly.

Question 1

(i) Let V be the vector space of $m \times n$-matrices over a field F. Fix an $m \times m$ matrix A and an $n \times n$-matrix C, and consider the map

$$\phi : V \longrightarrow V$$

defined by

$$\phi(B) = ABC.$$

Show that ϕ is linear and determine the trace of ϕ in terms of A and C.

(ii) Determine the determinant of ϕ in terms of A and C.

(iii) Consider a system $\Sigma(y)$ of m linear equations in n variables x_1,..., x_n:

$$\sum_{j=1}^{n} a_{ij}(y) \cdot x_j = b_i(y), \quad i = 1, 2, ..., m,$$

whose coefficients $a_{ij}(y)$ and $b_i(y)$ are polynomials over $\mathbb{C}$ in a variable y. Suppose that for some $y_0 \in \mathbb{C}$, $\Sigma(y_0)$ has a unique solution. Show that $n \leq m$ and that for all but finitely many $y \in \mathbb{C}$, the system $\Sigma(y)$ has at most one solution.

(iv) If, further $m = n$, show that $\Sigma(y)$ has a unique solution for all but finitely many $y \in \mathbb{C}$.

Question 2

Let G be a finite abelian group. A character of G is a group homomorphism

$$\chi : G \longrightarrow \mathbb{C}^\times,$$

where $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers.

(i) If $\chi_1, ..., \chi_r$ are distinct characters of G, show that $\chi_1, ..., \chi_r$ are linearly independent elements in the vector space of $\mathbb{C}$-valued functions on G. Deduce that G has at most $|G|$ distinct characters.

(ii) Show that G has exactly $|G|$ distinct characters. (Hint: you may want to use the fundamental theorem of finite abelian groups).

(iii) Let $\widehat{G}$ denote the set of all characters of G. Show that $\widehat{G}$ is a group under the product:

$$(\chi_1 * \chi_2)(g) = \chi_1(g) \cdot \chi_2(g) \quad \text{for all } g \in G.$$

(iv) Is it true that $\widehat{G}$ is isomorphic to G as groups? Justify your answer.

Question 3

(i) Give the definition of a Euclidean domain and a principal ideal domain.

(ii) Prove that a Euclidean domain is a principal ideal domain.

(iii) Decide with justification whether $\mathbb{Z}[x]$ is a principal ideal domain.

(iv) Decide with justification whether $\mathbb{Z}[\sqrt{-2}] = \mathbb{Z}[x]/(x^2 + 2)$ is a principal ideal domain.

Question 4

(i) Let p be an odd prime. Show that the polynomial $f(x) = x^p - 2$ is irreducible over $\mathbb{Q}$.

(ii) Determine the splitting field K of $f(x)$ in $\mathbb{C}$ in the form $\mathbb{Q}(x, y)$ for some elements $x, y \in \mathbb{C}$.

(iii) Determine the Galois group of K over $\mathbb{Q}$ by writing down how its elements act on the generators x and y of K over $\mathbb{Q}$ arising in (ii).

(iv) Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{F}_p, \ a \neq 0,$$

where $\mathbb{F}_p$ denotes the finite field with p elements.

Question 5

Let R be a commutative ring with identity element and let M and N be R-modules.

(i) Give the definition of the R-module $M \otimes_R N$.

(ii) If $N_1 \longrightarrow N_2$ is injective, is the induced map $M \otimes_R N_1 \longrightarrow M \otimes_R N_2$ necessarily injective? Justify your answer.

(iii) If $N_1 \longrightarrow N_2$ is surjective, is the induced map $M \otimes_R N_1 \longrightarrow M \otimes_R N_2$ necessarily surjective? Justify your answer.

(iv) Show that

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

for R-modules M, N and R.