NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2011-2012

MA3227 Numerical Analysis II

April 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Each question in Section A carries 15 marks.
3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Calculators are not allowed to use during the examination.

SECTION A

Answer **ALL** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 marks]

Suppose that A is an $N \times N$ real-valued tri-diagonal matrix defined as follows,

$$A = \begin{pmatrix} p+q & q/2 & & & & \\ p/2 & p+q & q/2 & & & \\ & p/2 & p+q & q/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & p/2 & p+q & q/2 \\ & & & & p/2 & p+q \end{pmatrix}.$$

(a) Let I denote the $N \times N$ identical matrix. Prove that if $\|A\|_2 < 1$, then $I - A$ is invertible and

$$\|(I-A)^{-1}\|_2 \leq \frac{1}{1 - \|A\|_2}.$$

(b) Suppose that $pq > 0$. Prove that the sequence produced by the Gauss-Jacobi method converges to the solution of $Ax = b$ for any starting vector.

Question 2 [15 marks]

Consider the function $f(x) = 2xe^x - e^x - 1$.

(a) If Newton's method is used to find a zero point of f starting with $x_0 = 1$, what is the value of x_1 after one iteration.

(b) Design a fixed-point method for finding a zero point of f in the interval $[0, 1]$. Justify its convergence.

Question 3 [15 marks]
Consider the following initial value problem:

$$\begin{cases} y'(t) = t^2 - 2y, \quad 0 \le t \le 1, \\ y(0) = 1. \end{cases} \tag{1}$$

(a) Solve the initial value problem (1) using the mid-point algorithm of order 2 with step size $h = \frac{1}{2}$.

(b) Prove that the local truncation error of the mid-point algorithm is of order 2.

Question 4 [15 marks]
Consider the two-point boundary-value problem:

$$\begin{cases} y'' + 2y' + y + t = 0, \quad 0 \le t \le 2, \\ y(0) = 0, \quad y(2) = 2. \end{cases} \tag{2}$$

(a) Solve (2) for $y(1)$ using the finite difference method with step size $h = 1$.

(b) Show how the linear shooting method can be applied to solve (2).

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Section B

Answer not more than **TWO** *questions in this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

Consider a well-posed initial value problem:

$$\begin{cases} y'(t) = f(t, y), \quad 0 \le t \le 1, \\ y(0) = y_0. \end{cases} \tag{3}$$

Suppose that f is infinitely differentiable on $\mathbb{R}^2$. For any given positive integer N, let $h = \frac{1}{N}$ and $t_i = ih$ for $i = 0, 1, \cdots, N$. Consider the following two step explicit method for solving (3):

$$\begin{cases} \omega_0 = y_0, \quad \omega_1 = y_1, \\ \omega_{i+1} = a\omega_i + (1-a)\omega_{i-1} + h(b_1 f(t_i, \omega_i) + b_2 f(t_{i-1}, \omega_{i-1})), \quad i = 1, 2, \cdots, N-1. \end{cases} \tag{4}$$

(a) Determine the values of a, b_1 and b_2 which maximize the order of the local truncation error of the method (4).

(b) Prove that the method (4) is stable if $|a - 1| < 1$.

Question 6 [20 marks]

Consider an infinitely differential function f defined on $\mathbb{R}$. Let ∇ denote the backward-difference operator defined by

$$\begin{cases} (\nabla^0 f)(x) = f(x), \\ (\nabla^n f)(x) = (\nabla^{n-1} f)(x) - (\nabla^{n-1} f)(x-1), \quad n = 1, 2, 3, \cdots. \end{cases}$$

(a) Consider a function y defined by

$$y(x) = \sum_{k=0}^{2} d_k \nabla^k f(x),$$

where $d_k = (-1)^k \int_{1-}^{0} \binom{-s}{k} ds$ for $k = 0, 1, 2$. Prove that

$$y(x) = \frac{1}{12}[5f(x) + 8f(x-1) - f(x-2)].$$

(b) For any polynomial g of degree no more than a positive integer m, show that

$$\nabla^{m+1} g = 0.$$

Question 7 [20 marks]

Let $x \in \mathcal{N}(0, \sigma^2)$ be a Gaussian random variable with probability density function p defined as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

(a) Let $\mathbb{E}[x]$ denote the expectation of x. Prove that

$$\mathbb{E}[e^{-x^2}] = \frac{1}{\sqrt{2\sigma^2 + 1}}.$$

(b) Let $P(x \geq \frac{5}{2}\sigma)$ denote the probability of $x \geq \frac{5}{2}\sigma$. Show how to calculate $P(x \geq \frac{5}{2}\sigma)$ using Monte Carlo sums.

END OF PAPER