NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2011-2012

MA2213 Numerical Analysis I

April 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 20 marks.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks. If you have answered all three questions, state precisely on the cover of the answer script which two questions should be marked. Otherwise, the two with the lowest marks will be used to compute the total marks.

4. This is a closed book exam. However, candidates are allowed to bring an A4 sized help sheet which can be written on both sides.

5. Only scientific calculator is allowed in the exam.

SECTION A

Answer **ALL** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [20 marks]

(a) Notice that we cannot directly apply Simpson's rule to estimate

$$I = \int_0^1 \frac{e^x}{\sqrt{x}} dx.$$

However, we can first compute $\int_0^1 \frac{1}{\sqrt{x}} dx$ by the fundamental theorem of calculus, and then apply Simpson's rule to estimate $\int_0^1 \frac{e^x - 1}{\sqrt{x}} dx$. Using this idea to obtain an estimation of I. You should use a scientific calculator and keep 4 significant digits in your final result.

(b) Suppose we want to approximate $I(f) = \int_{-1}^1 f(x)dx$ by

$$I_4(f) = A_1 f(-1) + A_2 f(x_1) + A_2 f(x_2) + A_1 f(1).$$

Determine the constants A_1, A_2, x_1, and x_2 so that the quadrature has the highest degree of precision with respect to f. Then determine the highest degree of precision with respect to f.

Question 2 [20 marks]

(a) Given data (x_k, f_k) for $k = 1, ..., n$ and the interpolating polynomial $p(x) \in P_{n-1}$, determine the a_i $(i = 1, ..., n)$ in the following expansion in terms of (x_k, f_k), $k = 1, ..., n$:

$$\frac{p(x)}{(x - x_1) \cdots (x - x_n)} = \sum_{i=1}^{n} \frac{a_i}{x - x_i}.$$

(b) Let

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin t) dt,$$

be the Bessel function of order zero and let $p_n(x)$ be the polynomial of degree n or less which interpolates $J_0(x)$ at $x_{n,i} = \frac{i}{n}$, $i = 0, 1, ..., n$. What is the behavior of the maximal interpolation error

$$\max_{0 \le x \le 1} |p_n(x) - J_0(x)|$$

as $n \to \infty$?

Question 3 [20 marks]

(a) [5 marks] Suppose $S, T \in \mathbb{R}^{n\times n}$ are upper triangular and that $(ST)\,x = b$ is a non-singular system. How to solve for x with only $O(n^2)$ multiplication/divisions? One writes $g_n = O(n^2)$ if and only if there is a positive constant M such that for all sufficiently large values of n, g_n is at most M multiplied by n^2 in absolute value. For example, $g_n = 4n^2 + 3n = O(n^2)$.

(b) [8 marks] Suppose

$$S_+ = \begin{bmatrix} \sigma & u^\top \\ 0 & S_c \end{bmatrix}, \qquad T_+ = \begin{bmatrix} \tau & v^\top \\ 0 & T_c \end{bmatrix}, \qquad I_+ = \begin{bmatrix} 1 & 0 \\ 0 & I_c \end{bmatrix}, \qquad b_+ = \begin{bmatrix} \beta \\ b_c \end{bmatrix},$$

where S_+, S_c, T_+, and T_c are square matrices, I_+ and I_c are identity matrices, b_+ is a column vector, and σ, τ, and β are real numbers. Show that if we have a vector x_c such that

$$(S_c T_c - \lambda I_c)\, x_c = b_c \tag{1}$$

and $w_c = T_c x_c$ is available, then

$$x_+ = \begin{bmatrix} \gamma \\ x_c \end{bmatrix} \qquad \text{with} \qquad \gamma = \frac{\beta - \sigma v^\top x_c - u^\top w_c}{\sigma\tau - \lambda} \tag{2}$$

solves $(S_+T_+ - \lambda I_+)\, x_+ = b_+$.

(c) [7 marks] Suppose $S, T \in \mathbb{R}^{n\times n}$ are upper triangular and that $(ST - \lambda I)\, x = b$ is a non-singular system, where λ is some constant and $I \in \mathbb{R}^{n\times n}$ is the identity matrix. How to solve for x with only $O(n^2)$ multiplication/divisions?

SECTION B

Answer not more than **TWO** *questions in this section. Each question in this section carries 20 marks.*

Question 4 [20 marks] Recall that given data points $\{(x_i, f_i) : i = 1, ..., n\}$ with $f_i = f(x_i)$ for some function f, we can construct an interpolating polynomial p_{n-1} of degree $n-1$ or less which satisfies $p_{n-1}(x_i) = f(x_i)$. Then given any number x, we can use $p_{n-1}(x)$ as an estimation of $f(x)$. Now, suppose our problem is to solve $f(x) = 0$ with $f(x) = x^2 - 4\sin^2(\pi x) + 3$ and suppose we are given the following table. To find the root with inverse interpolation, we may reverse the roles of x_i and f_i and view the f_i numbers as arguments and the x_i numbers as values. Use this idea to find an estimation of the root of

$$x^2 - 4\sin^2(\pi x) + 3 = 0$$

on the interval $[0, \frac{1}{2}]$. Keep 3 significant digits in your final result.

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
$f(x) = x^2 - 4\sin^2(\pi x) + 3$	3	$\frac{73}{36}$	$\frac{1}{9}$	$-\frac{3}{4}$

Question 5 [20 marks]

(a) Prove or disprove

$$v \in \mathbb{R}^n \Rightarrow \|v\|_1 \|v\|_\infty \leq \sqrt{n}\|v\|_2.$$

(b) Recall that the ∞-norm condition number of a non-singular matrix A is defined as $\text{cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$. Construct a sequence $\{A_n \,|\, A_n \in \mathbb{R}^{n\times n}, n = 1, 2, 3, ...\}$ which has the following properties and then justify your answer.

$$\lim_{n\to\infty} \text{cond}_\infty(A_n) = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \det(A_n) = 1.$$

Construct a sequence $\{B_n \,|\, B_n \in \mathbb{R}^{n\times n}, n = 1, 2, 3, ...\}$ which has the following properties and then justify your answer.

$$\lim_{n\to\infty} \text{cond}_\infty(B_n) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \det(B_n) = 0.$$

Question 6 [20 marks]

(a) Find the exact value of the Discrete Fourier Transform (DFT) of $[1, 0, 0, 0, -2, -1, 0, 0]^\top$.

(b) Find the least square trigonometric approximating functions of orders 4, 6 and 8 for the following data points:

t	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
y	3	2	1	0	-1	0	1	2

Keep 3 significant digits for your final result. You may use the fact that the DFT of $x = [3, 2, 1, 0, -1, 0, 1, 2]^\top$ is

$$\begin{bmatrix} 2.8284 \\ 2.4142 \\ 0 \\ 0.4142 \\ 0 \\ 0.4142 \\ 0 \\ 2.4142 \end{bmatrix}$$

END OF PAPER