NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION, AY 2011/2012

MA2101 Linear Algebra II

May 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions.
3. Marks for each question are indicated at the beginning of the question. The marks for questions are not necessarily the same.
4. Calculators may be used. However, various steps in the calculations should be systematically laid out.

Question 1 [15 marks]

Let W_1 and W_2 be vector subspaces of a vector space V over a field F.

(a) Show that the sum $W_1 + W_2$ is a vector subspace of V.

(b) Assume that $\dim W_i = n_i < \infty$ ($i = 1$ and 2). Use the second isomorphism theorem or otherwise, to show that the following two conditions are equivalent.

(bi) The sum $W_1 + W_2$ is a direct sum.

(bii) $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$.

(c) Without the assumption $\dim W_i < \infty$, are (bi) and (bii) always equivalent? If the answer is 'yes', prove so; if the answer is 'no', provide a *concrete* counterexample.

Question 2 [15 marks]

(a) Let $A \in M_{m\times n}(\mathbf{R})$ and $B \in M_{n\times r}(\mathbf{R})$ be real matrices. Let

$$T_A : \mathbf{R}_c^n \to \mathbf{R}_c^m, \quad T_B : \mathbf{R}_c^r \to \mathbf{R}_c^n$$

be the associated linear transformations. You may use the fact that

$$\text{rank}(A) = \dim T_A(\mathbf{R}_c^n), \quad \text{rank}(AB) = \dim T_{AB}(\mathbf{R}_c^r) = \dim T_A(T_B(\mathbf{R}_c^r)).$$

Use the first isomorphism theorem or otherwise, to show that

$$\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\} \le \min\{m, n, r\}.$$

(b) Let $T : V \to W$ be a linear transformation between *finite*-dimensional vector spaces over the same field $\mathbf{R}$ of real numbers. Let

$$B_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}, \quad B_W = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$$

be bases of V and W, respectively. Show that the following three statements are equivalent (**Note.** $n = m$ is *not* assumed).

(bi) T is an isomorphism.

(bii) There is a linear transformation $S : W \to V$ such that the compositions satisfy:

$$S \circ T = I_V, \quad T \circ S = I_W.$$

(biii) The representation matrix $[T]_{B_V, B_W}$ is a *square* matrix and *invertible*.

Question 3 [10 marks]

Let $A \in M_6(\mathbf{R})$ be a real matrix with $m(x) := m_A(x) = (x-1)^2(x-2)$ the minimal polynomial.

(a) Determine which of the following polynomials

$$p_1(x) = (x-1)^4(x-2)^2,\ p_2(x) = (x-1)^3(x-2)^2(x-3),\ p_3(x) = (x-1)^2(x-2)^3$$

can be the characteristic polynomial $p_A(x)$ of A. Justify your answer(s).

(b) Find all *possible* Jordan canonical forms of A with respect to each of your answer(s) in (a). Justify your answers.

Question 4 [15 marks]

Let W be a vector space defined over the field $\mathbf{R}$ of real numbers and let

$$B_1 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

be a basis of W. Let $T : W \to W$ be a linear operator with

$$K := [T]_{B_1}$$

the representation matrix relative to the basis B_1.

(a) Assume that

$$Kq_1 = q_1,\ Kq_2 = 2q_2,\ Kq_3 = 3q_3$$

where

$$(q_1\ q_2\ q_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(ai) Find three linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of T. Express each $\mathbf{v}_i$ as a linear combination of elements in B_1.

(aii) Prove that $B_2 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of W.

(b) Let

$$P = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and define vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ such that

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)P.$$

(bi) Prove that $B_3 := \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of W.

(bii) Find an invertible matrix Q such that $Q\,[T]_{B_3}\,Q^{-1}$ is equal to a diagonal matrix.

Question 5 [10 marks]

Let $H \in M_4(\mathbf{R})$ be a real matrix. Assume that

$$(H^2 - I_4)(H + 2I_4) = 0.$$

(a) Find *all* possible minimal polynomials $m(x) := m_H(x)$ and characteristic polynomials $p(x) := p_H(x)$ of H. Justify your answers.

(b) Is H diagonalizable? Justify your answer.

(c) Is H an invertible matrix? If the answer is 'no', prove so; if the answer is 'yes', with respect to each of your answers in (a), find a corresponding polynomial $f(x)$ such that $H^{-1} = f(H)$.

Question 6 [15 marks]

Consider the real matrix $A \in M_3(\mathbf{R})$ below:

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(a) Find all eigenvalues of A.

(b) For each eigenvalue λ of A, find an orthonormal basis of the eigenspace $V_\lambda(A)$.

(c) Find an orthogonal matrix P such that $P^t AP$ is equal to a diagonal matrix.

Question 7 [10 marks]

Let $A \in M_n(\mathbf{C})$ $(n \geq 1)$ be a positive definite matrix.

(a) Show that $A = GG^*$ for some *invertible* matrix G.

(b) Can you write $A = E^2$ for some *self-adjoint and invertible* matrix E? Justify your answer.

(c) Assume that $L \in M_n(\mathbf{C})$ is a self-adjoint invertible matrix. Is L^2 positive definite? Justify your answer.

Question 8 [10 marks]

Let $(V, \langle , \rangle)$ be a finite-dimensional complex inner product space. Let $T : V \to V$ be a linear operator and $T^* : V \to V$ the adjoint of T. For a vector subspace Y of V, denote by

$$Y^{\perp} := \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{y} \rangle = 0, \ \forall \, \mathbf{y} \in Y\}.$$

(a) Assume that W is a T^*-invariant vector subspace of V. It is known that $W^{\perp}$ is a vector subspace of V. Show that $W^{\perp}$ is a T-invariant subspace of V.

(b) Assume that U is a T-invariant vector subspace of V. Is $U^{\perp}$ a T-invariant subspace of V? If the answer is 'yes', prove so; if the answer is 'no', provide a *concrete* counterexample.

END OF PAPER