NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER II EXAMINATION 2011-2012

MA1104 Multivariable Calculus

April 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-size help sheets into the examination room.

2. This examination paper contains a total of **NINE (9)** questions and comprises **SIX (6)** printed pages.

3. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [12 marks]

Let P be the point $(11, -3, 16)$ and let L be the line $\mathbf{r}(t) = \langle 2, 3, -1\rangle + t\langle 3, -4, 3\rangle$.

(i) Let

$$ax + by + cz = 95$$

be the equation of the plane which contains P and L.
Determine a, b, c.

(ii) Find the vector equation of the line which passes through P and intersects L at 90 degrees. Express your answer in the form

$$\mathbf{s}(t) = \langle 11, -3, 16\rangle + t\langle \alpha, \beta, \gamma\rangle$$

where $\langle \alpha, \beta, \gamma\rangle$ is a unit vector.

(iii) Find the distance d from P to the line L.

(iv) Find the point Q on the line L such that the length of PQ is d in (iii).

Question 2. [15 marks]

Let $\mathbf{r}(t)$ be a vector valued function for all $t \in \mathbb{R}$. Suppose for all $t \in \mathbb{R}$, all higher derivatives of $\mathbf{r}(t)$ exist and $\mathbf{r}'(t) \neq \langle 0, 0, 0\rangle$.

We set $v(t) = |\mathbf{r}'(t)|$, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{v(t)}$ and the curvature $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{v(t)}$.

(i) Show that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$ for all $t \in \mathbb{R}$.

(ii) Show that

$$|\mathbf{T}(t) \times \mathbf{T}'(t)| = |\mathbf{T}'(t)|$$

for all $t \in \mathbb{R}$.

(iii) Show that

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = v(t)^n\ \kappa(t)$$

for some integer n.
Determine the value of n.
(Hint: First express $\mathbf{r}'(t) \times \mathbf{r}''(t)$ in terms of $v(t)$, $\mathbf{T}(t)$ and $\mathbf{T}'(t)$.)

(iv) Consider graph $y = x^6$ on the xy-plane. Using (iii) or otherwise, compute its curvature at $(1, 1)$.

Question 3. [10 marks]

State clearly if the following limits exist:

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{12x^3y^5 + 4x^4y^4}{x^6 + 4y^8}$,

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{12x^3y^4 + 4x^4y^4}{x^6 + 4y^8}$.

If the limit does not exist, give a proof using the two paths test.
If the limit exists, compute its value.

Question 4. [13 marks]

Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(i) Is $f(x,y)$ continuous at $(0,0)$? Justify your answer.

(ii) Does $f_x(0,0) = \dfrac{\partial f}{\partial x}(0,0)$ exist?
If it exists, compute its value.
If it does not exist, give a proof.

(iii) Does $f_{xy}(0,0) = \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ exist?
If it exists, compute its value.
If it does not exist, give a proof.

(iv) Is $f(x,y)$ differentiable at $(0,0)$? Justify your answer.

Remark. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is a *differentiable* function at $(0,0)$ if the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, and there exist functions $\epsilon_1(x,y)$ and $\epsilon_2(x,y)$ such that

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + x\epsilon_1(x,y) + y\epsilon_2(x,y)$$

where $\displaystyle\lim_{(x,y)\to(0,0)} \epsilon_1(x,y) = \lim_{(x,y)\to(0,0)} \epsilon_2(x,y) = 0$.

Question 5. [8 marks]

(i) Find the tangent plane of the surface

$$x^2 z + 2y^2 + 2z^5 = 5$$

at the point $P(1, -1, 1)$.
Express your answer in the form $ax + by + cz = d$.

(ii) Using linear approximation or otherwise, approximate the value of z that is close to 1 when $(x, y) = (1.05, -0.99)$.
Express your answer to the nearest 4 decimal places.

Question 6. [10 marks]

Let E denote the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{4^2} = 1$$

and let L denote the plane

$$12x + 8y + 9z = 51.$$

(i) Determine the point on E which is furthest away from L. Compute the distance.

(ii) Determine the point on E which is closest to L. Compute the distance.

Question 7. [12 marks]

Let

$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq \pi^2\}$$

denote the ball of radius π.

(i) Using the change of coordinates

$$\begin{aligned} x &= \frac{1}{3}u - \frac{2}{3}v + \frac{2}{3}w \\ y &= \frac{2}{3}u - \frac{1}{3}v - \frac{2}{3}w \\ z &= \frac{2}{3}u + \frac{2}{3}v + \frac{1}{3}w \end{aligned}$$

or otherwise, compute the triple integral

$$\iiint_B \cos\left(\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z\right) dxdydz.$$

(ii) Let $\langle a_1, a_2, a_3 \rangle$ be a unit vector. What is the value of

$$\iiint_B \cos\left(a_1 x + a_2 y + a_3 z\right) dxdydz$$

in terms of a_1, a_2, a_3?
There is NO need to show your actual computation.
Briefly describe the steps you would take to arrive at your answer.

Hint: For (i) you may assume that the matrix

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

is a rotation about an axis and

$$A^{-1} = A^{\top} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

Question 8. [10 marks]

Let $\mathbf{F} = \langle z^3 - 2y, 3x^2 - 4y, z + 3y \rangle$ be a vector field.

(i) Compute $\nabla \times \mathbf{F}$.

(ii) Let C denote the closed curve $\mathbf{r}(t) = \langle 3\cos 2t, -3\sin 2t, 2 \rangle$ for $0 \leq t \leq 2\pi$. Compute the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Question 9. [10 marks]

Let S be a closed orientable surface in $\mathbb{R}^3$.

We suppose that S is the boundary surface of an open, connected, bounded and simply connected region B in $\mathbb{R}^3$.

Let $\mathbf{n}$ denote the normal **unit** vector on S which is outward pointing.

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function.

(i) Show that

$$\oiint_S \mathrm{D}_{\mathbf{n}}\phi \; d\sigma = \iiint_B \triangle\phi \; dV$$

where $\mathrm{D}_{\mathbf{n}}\phi$ denote the directional derivative of ϕ along the unit vector $\mathbf{n}$ and

$$\triangle\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

(ii) Suppose $\triangle\phi = 0$. Show that

$$\oiint_S \phi(\mathrm{D}_{\mathbf{n}}\phi) \; d\sigma = \iiint_B |\nabla\phi|^2 \; dV.$$

END OF PAPER