NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION (2011–2012)

MA1102R Calculus

April/May 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [8 marks]

Let $f(x) = (x^2 + 2x + 2)e^{-x}$.

(i) Find the open intervals on which f is increasing and decreasing.

(ii) Find the coordinates of its local maximums and local minimums (if any).

(iii) Find the open intervals on which f is concave up and concave down.

(iv) Find the coordinates of its inflection points (if any).

Question 2 [12 marks]

Find the following limits.

(a) $\displaystyle\lim_{x\to 0^+}\left(\sqrt{x^3+x^2+x}\,\sin\frac{\pi}{x}\right)$.

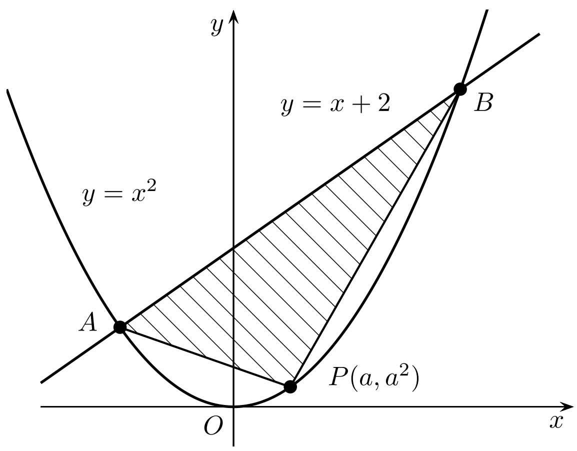
(b) $\displaystyle\lim_{x\to\infty}\left(\frac{x+\pi}{x+e}\right)^x$.

Question 3 [10 marks]

Let A and B be the points of interception of the parabola $y = x^2$ and the straight line $y = x + 2$. Let $P(a, a^2)$ be a point on the parabola between A and B.

(i) Show that the area of the triangle $\triangle ABP$ is given by $A(a) = \dfrac{3}{2}\,(a - a^2 + 2)$.

(ii) Find the coordinates of P so that the triangle $\triangle ABP$ has the largest area.



Question 4 [21 marks]

Find the following integrals.

(a) $\displaystyle\int x^3 \sin x \, dx.$

(b) $\displaystyle\int \frac{1}{(x^2+9)^{3/2}} \, dx.$

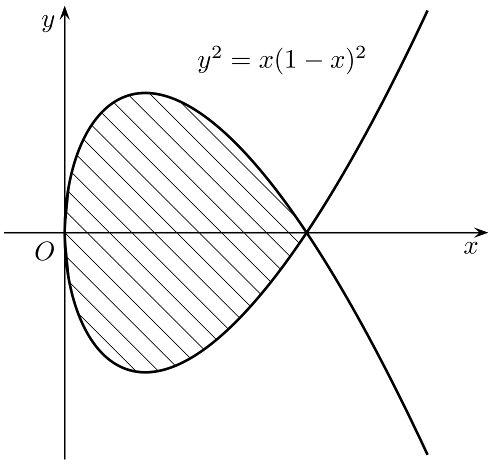
(c) $\displaystyle\int \frac{6x-2}{x^4-1} \, dx.$

Question 5 [12 marks]

Consider the region enclosed by the graph of $y^2 = x(1-x)^2$, as shown below.

(a) Find the volume of the solid that is generated if the region is revolved about the x-axis.

(b) Find the volume of the solid that is generated if the region is revolved about the y-axis.



Question 6 [12 marks]

(a) Let $p > 0$. Using the Riemann sum or otherwise, evaluate the following limit:

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}.$$

(b) If $F(x) = \displaystyle\int_2^{2\sqrt{x}} \left[\int_{16}^{t^4} \frac{\sqrt{1+u^4}}{u} \, du \right] dt$, find $F''(1)$.

Question 7 [15 marks]

(a) Solve the initial value problem

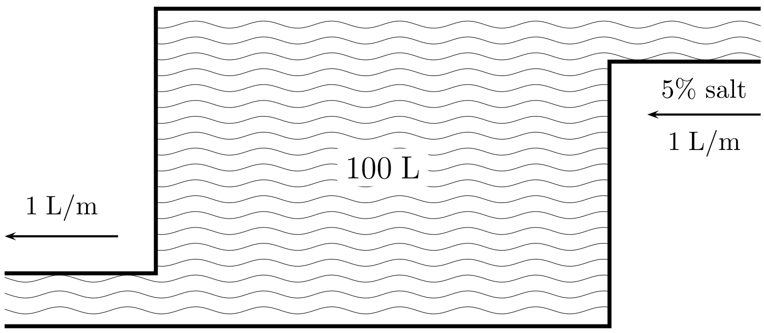
$$x^2 \frac{dy}{dx} + y - 2xy - x^2 = 0, \qquad y = 2 \text{ when } x = 1.$$

(b) A tank initially holds 100 liters of pure water. Brine that contains 5% of salt (in volume) enters the tank at the rate of 1 liter per minute, and the well-stirred mixture leaves at the same rate. Let $S = S(t)$ denote the amount of salt in the tank at time t.

(i) Show that S satisfies the ordinary differential equation

$$\frac{dS}{dt} = 0.05 - 0.01S.$$

(ii) Find the explicit expression of S in terms of t.



Question 8 [10 marks]

(a) Let f be a function continuous on $[a, b]$ and differentiable on (a, b), where $a < b$.

Suppose $f(a) = f(b)$. Prove that there exist numbers $c_1, c_2, \ldots, c_{2012} \in (a, b)$ satisfying $c_1 < c_2 < \cdots < c_{2012}$ and

$$f'(c_1) + f'(c_2) + \cdots + f'(c_{2012}) = 0.$$

(b) Let f be a twice differentiable function defined on $\mathbb{R}$ such that $f''(x) > 0$ for all $x \in \mathbb{R}$.

Prove that for any numbers $x_1, x_2, \ldots, x_{2012} \in \mathbb{R}$,

$$f\left(\frac{x_1 + x_2 + \cdots + x_{2012}}{2012}\right) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_{2012})}{2012}.$$

End of Paper