Student Number:

A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2011-2012

MA1101R LINEAR ALGEBRA I

April/May 2012 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **FOUR (4)** questions and comprises **NINETEEN (19)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
Total	

Question 1 (a) [15 marks]

Let $\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \boldsymbol{u_4} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}.$

(i) (3 marks) Show that $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbb{R}^3$.

(ii) (4 marks) Find the coordinate vector $[\boldsymbol{u_4}]_S$ with respect to S.

(iii) (3 marks) Prove that for all $k \in \mathbb{R}$, $[k\boldsymbol{u_4}]_S = k\,[\boldsymbol{u_4}]_S$.

(iv) (5 marks) Find a basis for $\text{span}\{\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ and determine its dimension.

Use the space below to write your answer and working

(More working spaces for Question 1 (a))

Question 1 (b) [5 marks]

The augmented matrix of a homogeneous linear system has the following reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & k_1 & 0 \\ 0 & 1 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \end{array}\right).$$

If the solution space of this system is span$\{\boldsymbol{u_3}\}$ where $\boldsymbol{u}_3$ is as in part (a), find k_1, k_2, k_3. Explain clearly how your answer is obtained.

Use the space below to write your answer and working

Question 1 (c) [5 marks]

Let S and $\boldsymbol{u}_4$ be as in part (a). Suppose $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is another basis for $\mathbb{R}^3$ such that

$$[\boldsymbol{u_4}]_T = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad [\boldsymbol{v_1}]_S = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}.$$

Find $\boldsymbol{v_3}$.

Use the space below to write your answer and working

Question 2 (a) [15 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$.

(i) (4 marks) Find a basis for the row space of $\boldsymbol{A}$. What is the rank of $\boldsymbol{A}$?

(ii) (3 marks) If $\boldsymbol{A}$ is the standard matrix for a linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$, determine whether $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ is in the range of T_1. Justify your answer.

(iii) (5 marks) If $\boldsymbol{A}^T$ is the standard matrix for a linear transformation $T_2 : \mathbb{R}^4 \to \mathbb{R}^3$, find a basis and determine the dimension of kernel of T_2.

(iv) (3 marks) Find two distinct vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ (that is, $\boldsymbol{v_1} \neq \boldsymbol{v_2}$) in the column space of $\boldsymbol{A}$ such that $T_2(\boldsymbol{v_1}) = T_2(\boldsymbol{v_2})$.

Use the space below to write your answer and working

(More working spaces for Question 2 (a))

Question 2 (b) [5 marks]

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \\ 0 & 1 & 1 & 0 \\ 1 & 0 & x+1 & -1 \end{pmatrix}$ and $\boldsymbol{C}$ be a 5×4 matrix of full rank.

Find all values of x such that $\boldsymbol{B}$ and $\boldsymbol{C}$ have the same row space.
Justify your answer.

Use the space below to write your answer and working

Question 2 (c) [5 marks]

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the matrices in part (a) and (b) respectively.
Show that for all values of x, the column space of $\boldsymbol{A}$ is a subset of the row space of $\boldsymbol{B}$.

Use the space below to write your answer and working

Question 3 (a) [15 marks]

(i) (4 marks) Find the characteristic polynomial of the symmetric matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

and show that the eigenvalues of $\boldsymbol{A}$ are 2 and 4.

(ii) (5 marks) Find a basis for each of the eigenspaces of $\boldsymbol{A}$.

(iii) (3 marks) Find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

(iv) (3 marks) Find a symmetric matrix $\boldsymbol{C}$ such that $\boldsymbol{C}^2 = \boldsymbol{A}$. (You may leave your answer as a product of matrices.)

Use the space below to write your answer and working

(More working spaces for Question 3 (a))

Question 3 (b) [5 marks]

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ be a basis for $\mathbb{R}^4$ and $\boldsymbol{B}$ a 4×4 matrix such that:

$$\boldsymbol{B}\boldsymbol{u}_1 = \boldsymbol{u}_2, \quad \boldsymbol{B}\boldsymbol{u}_2 = \boldsymbol{u}_1, \quad \boldsymbol{B}\boldsymbol{u}_3 = \boldsymbol{u}_4, \quad \boldsymbol{B}\boldsymbol{u}_4 = \boldsymbol{u}_3.$$

Find <u>all</u> eigenvalues of $\boldsymbol{B}$ and determine whether $\boldsymbol{B}$ is diagonalizable.
Justify your answers.

Use the space below to write your answer and working

Question 3 (c) [5 marks]

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two diagonalizable 3×3 matrices, both having exactly two eigenvalues 1 and -1.
Suppose 2 and -2 are not eigenvalues of $\boldsymbol{A} + \boldsymbol{B}$. Show that $\boldsymbol{A} + \boldsymbol{B}$ is singular.

Use the space below to write your answer and working

Question 4 (a) [15 marks]

Let $\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\boldsymbol{u}_3 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

(i) (3 marks) Show that the subspace $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ of $\mathbb{R}^3$ is orthogonal to $\boldsymbol{u}_3$.

(ii) (3 marks) Find an orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for V such that $\boldsymbol{v}_1$ is parallel to $\boldsymbol{u}_1$.

(iii) (3 marks) Find the projection of $\boldsymbol{w}$ onto V.

(iv) (3 marks) Find the equation of a plane that is perpendicular to V and contains $\boldsymbol{w}$.

(v) (3 marks) Write down two orthogonal matrices both having $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as its first two columns respectively.

Use the space below to write your answer and working

(More working spaces for Question 4 (a))

Question 4 (b) [5 marks]

Find the least squares solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Use the space below to write your answer and working

Question 4 (c) [5 marks]

Show that every invertible matrix $\boldsymbol{A}$ can be written as $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}$ where $\boldsymbol{B}$ is an orthogonal matrix and $\boldsymbol{C}$ is an upper triangular matrix.

Use the space below to write your answer and working

(More working spaces. Please indicate the question numbers clearly.)

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[END OF PAPER]