NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2011-2012

MA2213 Numerical Analysis I

November 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**. Each question in Section A carries 20 marks.
3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks. If you have answered all three questions, state precisely on the cover of the answer script which two questions should be marked. Otherwise, the two with the lowest marks will be used to compute the total marks.
4. This is a closed book exam. However, candidates are allowed to bring an A4 sized help sheet which can be written on both sides.
5. Candidates should do all the computations exactly and do not do any round off unless otherwise stated in the question. Candidates should lay out systematically the various steps in the calculations.

SECTION A

Answer **ALL** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [20 marks]

(a) Use Simpson's rule to estimate $\int_{-1}^{1}|x|\frac{1}{1+x^4}$ and $\int_{-1}^{1}|x|\frac{1}{|\sin(\pi x/2)|(1+x^4)}$.

(b) Suppose we want to approximate $\int_{-1}^{1}|x|f(x)dx$ by some quadrature $A\left(f(x_1)+f(x_2)\right)$. Determine the constants A, x_1, and x_2 so that the quadrature has the highest degree of precision with respect to f. Then determine the highest degree of precision with respect to f.

Question 2 [20 marks]

(a) Given $x=(\frac{1}{4},-\frac{1}{2},\frac{1}{2},1)^\top$, find $F\in\mathbb{C}^{4\times 4}$ so that the discrete Fourier transform of x can be written as Fx. Compute Fx and then determine the smallest integer k so that $F^k x \overset{define}{=\!=} \underbrace{F(F(F(\cdots Fx)))}_{k\,\text{times}} = x$. [Hint: Note that F is symmetric and unitary. What is FF?]

(b) Find the trigonometric interpolant in $\mathcal{T}$ for the data points $\left(0,\frac{1}{4}+2\sqrt{2}\right)$, $\left(\frac{1}{4},-\frac{1}{2}+2\sqrt{2}\right)$, $\left(\frac{1}{2},\frac{1}{2}+2\sqrt{2}\right)$, $\left(\frac{3}{4},1+2\sqrt{2}\right)$. Here $\mathcal{T}=\text{span}\{1,\cos(2\pi t),\sin(2\pi t),\cos(4\pi t)\}$.

Question 3 [20 marks]

For certain function $f(x)$, we know the values of $f(a)$, $f(a-h)$, and $f(a-2h)$, where a and h are given numbers. Let $p_2(x)$ be the quadratic polynomial that interpolates $f(x)$ at points a, $a-h$, and $a-2h$.

(a) Write down the formula of $p_2(x)$ and then find the value of $\frac{d}{dx}p_2(x)$ at $x=a$.

(b) Determine the integer k such that $\frac{dp_2}{dx}(a)$ is a k-th order approximation of $f'(a)$ when $h\to 0$. Justify your answer using the Taylor's expansion. Here you can assume f is infinitely differentiable on the interval $(a-1,a+1)$.

SECTION B

Answer not more than **TWO** *questions in this section. Each question in this section carries 20 marks.*

Question 4 [20 marks]

(a) For certain function $f(x)$, we know $f[0] = 1$, $f[0,1] = -1$, $f[0,1,2] = 2$. Furthermore, we know the absolute value of $f[0,1,2,x]$ is less than or equal to 3 for any $x \in [0,1]$.

Determine the quadratic polynomial $p_2(x)$ that interpolates $f(x)$ at $x = 0, 1, 2$. Then find a good upper bound for $|f(0.5) - p_2(0.5)|$.

(b) Consider the function $\sin(\pi x)$ on $[-1,1]$ and its approximations by interpolating polynomials. For integer $n \geq 1$, let $x_{n,j} = -1 + \frac{2j}{n}$ for $j = 0, 1, \cdots, n$, and let $p_n(x)$ be the nth-degree polynomial interpolating $\sin(\pi x)$ at the nodes $x_{n,0}, \cdots, x_{n,n}$. Prove that

$$\max_{x \in [-1,1]} |\sin(\pi x) - p_n(x)| \to 0 \qquad \text{as} \quad n \to \infty.$$

Question 5 [20 marks]

(a) Find the $PA = LU$ factorization of the matrix $A = \begin{bmatrix} \frac{1}{2} & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(b) Assume A is a general $n \times n$ matrix with entries $|a_{ij}| \leq 1$ for $1 \leq i, j \leq n$. Prove that the matrix U in the $PA = LU$ factorization satisfies $|u_{ij}| \leq 2^{n-1}$ for all $1 \leq i, j \leq n$. You can use words to describe the steps for the $PA = LU$ decomposition. And you can use words to describe how the entries of U gradually change during the procedure. However, your statement must be relevant, clean, and precise in order to get credit.

Question 6 [20 marks]

(a) Find the line $y = a + bt$ that best fits the data points $(-3,3), (-1,2), (0,0), (1,-1)$ in the least squares sense.

(b) Let $t_j = j/100$, $a_j = j$, $b_j = -j$, for $j = 0, 1, 2, ..., 99$. Define

$$f(t) = \sum_{k=0}^{99} \left(a_k \cos(2\pi kt) + b_k \sin(2\pi kt)\right). \tag{1}$$

Denote $f(t_j)$ by x_j for $j = 0, ..., 99$. Determine the values of c_ℓ, d_m for $\ell = 0, ..., 5$, $m = 1, ..., 4$, so that

$$P(t) = c_0 + \sum_{k=1}^{4} \left(c_k \cos(2\pi kt) + d_k \sin(2\pi kt)\right) + c_5 \cos(10\pi t)$$

is the least squares approximation to the data points (t_j, x_j) for $j = 0, ..., 99$. [Hint: Recall for any integer k and for any $j = 0, ..., 99$, $\cos(2\pi(100-k)t_j) = \cos(2\pi kt_j)$, $\sin(2\pi(100-k)t_j) = -\sin(2\pi kt_j)$, and $\sin(2\pi 50 t_j) = 0$.]

END OF PAPER