NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

MA2108 Mathematical Analysis I

November 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.

3. Answer **ALL** questions in **Section A**. Section A carries a total of 70 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Section B carries a total of 30 marks.

5. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 70 marks.*

Question 1.

The sequence (a_n) is defined by

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{2}\left(1 + \frac{7}{a_n + 2}\right) \qquad \text{for all } n \in \mathbb{N}.$$

Prove that (a_n) converges and find its limit. [8 marks]

Question 2.

Let

$$x_n = \frac{(3 + (-1)^n)\,(n^3 + 1)\cos\left(\frac{n\pi}{6}\right)}{n(2n+1)(3n+2)}, \qquad n \in \mathbb{N}.$$

(i) Show that (x_n) is bounded. [2 marks]

(ii) Find $\limsup x_n$ and $\liminf x_n$. [6 marks]

(iii) Is (x_n) convergent? Justify your answer. [2 marks]

Question 3.

(a) Test the following series for convergence.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^{n+1}}\left(1 + \frac{1}{1+4n}\right)^{2n^2}$. [4 marks]

(ii) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{1+n^4} - n^2\right)$. [4 marks]

(b) Let (a_n) be a sequence.

(i) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})$ also converges.

[3 marks]

(ii) Give an example of (a_n) with the property that $\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})$ converges but $\sum_{n=1}^{\infty} a_n$ diverges.

[3 marks]

(iii) Prove that if $\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})$ converges and $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

[6 marks]

Question 4.

(a) Use the $\varepsilon-\delta$ definition of limit to prove that

$$\lim_{x\to 1}\frac{x+2}{3x-2}=3.$$

[6 marks]

(b) In each part, either evaluate the limit or show that the limit does not exist.

(i) $\lim_{x\to 0}\left|\sin\left(\frac{1}{x^2}\right)\right|$. [4 marks]

(ii) $\lim_{x\to 3^+}\frac{[x]+1}{[5-x]+x^2}$.

Here $[t]$ denotes the greatest integer less than or equal to t. [4 marks]

(c) The functions f and g are defined in a deleted neighborhood of the point $x=a$. Prove that if $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, then

$$\lim_{x\to a}(f+g)(x)=\infty.$$

[6 marks]

Question 5.

(a) Prove that the function

$$f(x) = x^3$$

is not uniformly continuous on $(0, \infty)$. [4 marks]

(b) Let

$$g(x) = \frac{x^3 \cos\left(\frac{\pi}{\sqrt{x}}\right)}{(x+1)^2}, \quad x \in (0, 1].$$

(i) Determine whether the limit $\lim_{x \to 0+} g(x)$ exists. [4 marks]

(ii) Is g uniformly continuous on $(0, 1]$? Justify your answer. [4 marks]

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 30 marks.*

Question 6.

(a) For each $n \in \mathbb{N}$, let

$$a_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

(i) Prove that (a_n) is a decreasing sequence. [4 marks]

(ii) Prove that $a_n > e$ for each $n \in \mathbb{N}$, where e is the Euler number. [4 marks]

(b) Let (x_n) be a bounded sequence. For each $n \in \mathbb{N}$, let $y_n = x_{2n}$ and $z_n = x_{2n-1}$. Prove that

$$\limsup x_n = \max(\limsup y_n, \limsup z_n).$$

[7 marks]

Question 7.

(a) Suppose that the function f is continuous on $[a, b)$ and the limit $L = \lim_{x \to b^-} f(x)$ exists.

(i) Given an example to show that f may not have an absolute maximum in $[a, b)$. [2 marks]

(ii) Prove that if there is an $x_0 \in [a, b)$ such that $f(x_0) > L$, then f has an absolute maximum in $[a, b)$. [4 marks]

(iii) If there is an $x_1 \in [a, b)$ such that $f(x_1) = L$, then does f necessarily have an absolute maximum in $[a, b)$? Justify your answer. [4 marks]

(b) The function $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let

$$h(\mathbb{R}) = \{h(x) : \ x \in \mathbb{R}\}$$

be the range of h. Prove that if $h(\mathbb{R})$ is not bounded above and not bounded below, then $h(\mathbb{R}) = \mathbb{R}$. [5 marks]

Question 8.

(a) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and

$$f\left(r + \frac{1}{n}\right) = f(r)$$

for any rational number r and natural number n.

(i) Prove that for any rational number r and natural numbers n and m,

$$f\left(r + \frac{m}{n}\right) = f(r).$$

[3 marks]

(ii) Prove that f is a constant function. [5 marks]

(b) The function $g : (a, c) \to \mathbb{R}$ has the following property: There exists $b \in (a, c)$ such that g is uniformly continuous on $(a, b]$ and on $[b, c)$. Prove that g is uniformly continuous on (a, c). [7 marks]

END OF PAPER