NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

MA2101 Linear Algebra II

November 2011 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination.
2. The examination contains a total of **SIX (6)** questions and comprises **THREE (3)** printed pages.
3. The examination carries a total of 100 marks. Answer **ALL** questions.

Question 1 [10 marks]

Label the following statements as true or false. You do NOT need to justify your answers.

(a) Every normal operator is self-adjoint.

(b) Any subset of a linearly independent set is linearly independent.

(c) Every linear operator has at least one eigenvalue.

(d) The sum of two diagonalizable $n \times n$ matrices is diagonalizable.

(e) Every linear operator T on an inner product space has an adjoint T^*.

Question 2 [30 marks]

(a) Show that a linear transformation $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ is one-to-one if and only if $\mathsf{N}(\mathsf{T})$ (the null space of T) is $\{\mathbf{0}\}$.

(b) Give an example of a linear operator T on $\mathbb{R}^2$ such that $\mathbb{R}^2 \neq \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$, where $\mathsf{R}(\mathsf{T})$ denotes the range of T.

(c) Let A be a $n \times n$ matrix whose characteristic polynomial splits. Show that the sum of all eigenvalues of A (possibly with multiplicities) is equal to $\operatorname{tr}(A)$.

(d) Consider $\mathbb{C}^2$ as a vector space over the field $\mathbb{R}$. Let $\mathsf{W} = \{(z, \overline{z}) \in \mathbb{C}^2 : z \in \mathbb{C}\}$. Prove that W is a subspace of the real vector space $\mathbb{C}^2$ and find (with explanation) a basis for W.

(e) Let $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ whose only eigenvalues are $1, 2, 3, 4, 5$ possibly with multiplicities. Find the rank of $A + I$.

Question 3 [15 marks]

Let T be the linear operator on $\mathsf{P}_2(\mathbb{R})$, which consists of polynomials with real coefficients having degree less than or equal to 2, defined by

$$\mathsf{T}(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

Show that T is diagonalizable. Moreover, find a diagonal matrix D and a basis β for V such that $[\mathsf{T}]_\beta$ is D.

Question 4 [10 marks]

Let $\mathsf{W} = \operatorname{span}(\{(1, -1, i)\}) \in \mathbb{C}^3$. Find orthonormal bases for W and $\mathsf{W}^\perp$ respectively.

Question 5 [15 marks] Let V be a finite-dimensional inner product space over $\mathbb{C}$.

(a) Let T be a self-adjoint operator on V. Prove that $\langle \mathsf{T}(\mathbf{v}), \mathbf{v}\rangle$ is a real number for all $\mathbf{v} \in \mathsf{V}$.

(b) Show that a linear operator T is self-adjoint and $\langle \mathsf{T}(\mathbf{v}), \mathbf{v}\rangle \geq 0$ for all $\mathbf{v} \in \mathsf{V}$ if and only if $\mathsf{T} = \mathsf{U}^*\mathsf{U}$ for some linear operator U.

Question 6 [20 marks]

Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear operator. We say that T is nilpotent if $\mathsf{T}^p = \mathsf{T}_0$ for some positive integer p, where T_0 is the zero operator.

(a) Show that T is nilpotent if and only if all eigenvalues of T are zero.

(b) The trace $\operatorname{tr}(\mathsf{T})$ of T is defined to be $\operatorname{tr}(A)$, where A is the matrix of T with respect to some basis β. Show that this is well-defined, i.e., this does not depend on the choice of basis β.

(c) Show that T is nilpotent if and only if all of $\operatorname{tr}(\mathsf{T}^n)$, where n is a positive integer, are zero.

(d) Is statement (a) true for finite-dimensional vector spaces over $\mathbb{R}$? Justify your answer.

(e) Is statement (a) true for infinite-dimensional vector spaces over $\mathbb{C}$? Justify your answer.

END OF PAPER