NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

MA1507 Advanced Calculus

November 2011 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is an open book examination. Candidates are allowed to refer to any learning material.

2. This examination paper consists of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions.

4. Candidates may use calculators. However, they should lay out systematically the crucial steps in the calculations.

Answer **ALL** *questions.*

Question 1. [15 marks]

(a) Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{4x^4+y^2}.$$

(b) Consider the surface given by

$$\frac{1}{z^2+x^2} + \frac{1}{z^2+y^2} = 1.$$

(i) Suppose z is implicitly defined as a function of x and y by the above equation. Calculate the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ when $x = y = z = 1$.

(ii) Compute the equation of the tangent plane to the surface at the point $(1, 1, 1)$.

Question 2. [15 marks]

(a) Find a set of parametric equations for the tangent line to the curve of intersection of the surface $x^2 + z^2 = 2$ and the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.

(b) Maximize the function $P(x, y, z) = 3xz + 6y$ subject to the constraint $x^2 + 2y^2 + z^2 = 6$.

Question 3. [15 marks]

(a) Evaluate the following iterated integral:

$$\int_0^1 \int_{\sqrt{x}}^1 \frac{1}{4+y^3}\, dy\, dx.$$

(b) Compute the volume of the solid bounded by the following surfaces:

$$y = 1 - z^2, \quad y = 2x^2 + z^2 - 1.$$

Question 4. [15 marks]

(a) Let $\mathbf{F} = \langle x + x^2 e^y, -2xe^y \rangle$, and let C be the part of the circle $x^2 + y^2 = 4^2$ in the first quadrant of the xy-plane. Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds$$

where $\mathbf{N} = \langle x, y \rangle$.

(b) Find the surface area of the part of the cylinder $x^2 + y^2 = 1$ that lies above the xy-plane but below the surface $z = y^2$.

Question 5. [20 marks]

Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F} = \left\langle y - x, x - y, 3z + x^2 - y^2 \right\rangle, \quad (x, y, z) \in \mathbb{R}^3.$$

Suppose E is the solid in the first octant bounded by the surfaces $y^2 + z^2 = 1$, $y = x$, $x = 0$ and $z = 0$. Let S denote the boundary surface of E with positive orientation $\mathbf{n}$.

(a) Compute the flux

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS$$

where S_1 is the part of the plane $y = x$ that belongs to S.

(b) Compute

$$\iiint_E \operatorname{div} \mathbf{F}\, dV.$$

(c) Compute the flux

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS$$

where S_2 is the part of the surface $y^2 + z^2 = 1$ that belongs to S.

Question 6. [20 marks]

Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x, y, z) = \left\langle e^{\sin x} + xyz,\ z^2 + 5,\ x^2 + y^2 \right\rangle, \quad (x, y, z) \in \mathbb{R}^3.$$

(a) Parameterize the straight line directed from the point $(0, 2, 0)$ to $(0, 0, 1)$.

(b) Evaluate the line integral $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the straight line directed from $(0, 2, 0)$ to $(0, 0, 1)$.

(c) Calculate curl $\mathbf{F}$.

(d) Evaluate the line integral $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ where C_2 consists of the straight line directed from $(0, 0, 1)$ to $(2, 0, 1)$ followed by the curve given by

$$\mathbf{r}(t) = \left\langle 2\cos t,\quad 2\sin t,\quad 1 - \sin^2 t \right\rangle, \quad 0 \le t \le \frac{\pi}{2}.$$

END OF PAPER