NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

MA1104 Multivariable Calculus

November 2011 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-sized two-sided help sheets into the examination room.

2. This examination paper consists of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions.

4. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Answer **ALL** *questions.*

Question 1. [10 marks]

Suppose that $z = f(x, y)$, where $f(x, y)$, $f_x(x, y)$ and $f_y(x, y)$ are differentiable functions of x and y. Suppose that $x = e^{2u}$, $y = uv$.

Compute $\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)$ in terms of u, v and the partial derivatives of f with respect to x, y (including those of higher order).

Question 2. [15 marks]

Let f be the function on $\mathbb{R}^2$ defined by

$$f(x, y) = \begin{cases} \frac{xy^5}{x^4+y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Compute each of the following partial derivatives if it exists; otherwise, explain why it does not exist.

(i) $f_x(0, 0)$, $f_y(0, 0)$

(ii) $f_{xy}(0, 0)$, $f_{yx}(0, 0)$

(b) Is the function $f(x, y)$ differentiable at $(0, 0)$? Justify your answer.

(c) Is the function $f_{xy}(x, y)$ continuous at $(0, 0)$? Justify your answer.

Question 3. [15 marks]

Find all the point(s) that minimizes the function

$$f(x, y, z) = \frac{(xy)^2}{2} + \frac{(yz)^2}{2} + \frac{(xz)^2}{2}$$

subject to the condition that $x^2 + y^2 + z^2 = 4$ and $x, y, z \geq 1$.

Question 4. [15 marks]

(a) An insect is moving anticlockwise around the ellipse $2(x-2)^2 + (y-1)^2 = 4$ on the xy-plane. The temperature function $T(x,y)$ (experienced by the insect) is given by

$$T(x,y) = x^2 - y^2, \quad (x,y) \in \mathbb{R}^2$$

where T is measured in degree Celsius and x, y are measured in meters. How fast is the temperature changing in degree Celsius per meter when the insect is at the point $(3, \sqrt{2}+1)$?

(b) By using a double integral, evaluate the integral

$$\int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)}\, dx.$$

Question 5. [15 marks]

(a) Rewrite the integral

$$\int_0^2 \int_0^{4-y^2} \int_0^{y/2} f(x,y,z)\, dx\, dz\, dy$$

as an iterated integral in the order of $dy\, dz\, dx$.

(b) Evaluate the line integral

$$\int_C (e^x + 6xy)\, dx + (8x^2 + \sin(y^2))\, dy$$

where C is the negatively-oriented boundary of the region D (on the xy-plane) bounded by the curve $y = \sqrt{1-5x^2}$ and the x-axis.

Question 6. [15 marks]

Consider the vector field

$$\mathbf{F} = \left\langle \frac{y}{x^2+y^2+z^2},\ \frac{-x}{x^2+y^2+z^2},\ -z \right\rangle, \quad (x,y,z) \neq (0,0,0).$$

(a) Suppose E is the solid bounded by the following planes

$$x = 1, \quad x = -1, \quad y = 1, \quad y = -1, \quad z = 1, \quad z = -1.$$

Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the boundary surface of E with positive orientation (i.e. outward-pointing normal).

(b) Compute the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the sphere $x^2 + y^2 + z^2 = 6^2$ with negative orientation (i.e. inward-pointing normal).

Question 7. [15 marks]

(a) Suppose

$$\mathbf{F} = \left\langle z(1 - 2x^2 - y^2),\ z(2x^2 + y^2),\ e^{\sin z} + y \right\rangle, \quad (x,y,z) \in \mathbb{R}^3.$$

Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the curve of intersection of the cylinder $2x^2 + y^2 = 1$ and the plane $z + y = 7$ oriented clockwise when viewed from above (i.e. from the positive z-axis).

(b) Let T be the surface obtained by rotating the circle in the yz-plane of radius 1 centered at $(0, 2, 0)$ about the z-axis. Find the surface area of T.

END OF PAPER