NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION (2011–2012)

MA1102R Calculus

November 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

Let $f(x) = 3x^4 - 8x^3 - 90x^2$.

(i) Find the open intervals on which f is increasing and decreasing.

(ii) Find the coordinates of all the local maximum and minimum points of f.

(iii) Find the open intervals on which f is concave up and concave down.

(iv) Find the coordinates of all the inflection points of f.

Question 2 [13 marks]

(a) Using the ϵ, δ-definition, prove the limit $\lim_{x \to -3} \sqrt{x^2 + 16} = 5$.

(b) Find the values of real numbers a and b for which the function

$$f(x) = \begin{cases} x^2 + ax, & \text{if } x \leq 1, \\ -x^2 + b, & \text{if } x > 1, \end{cases}$$

is differentiable at $x = 1$. Justify your answers.

Question 3 [11 marks]

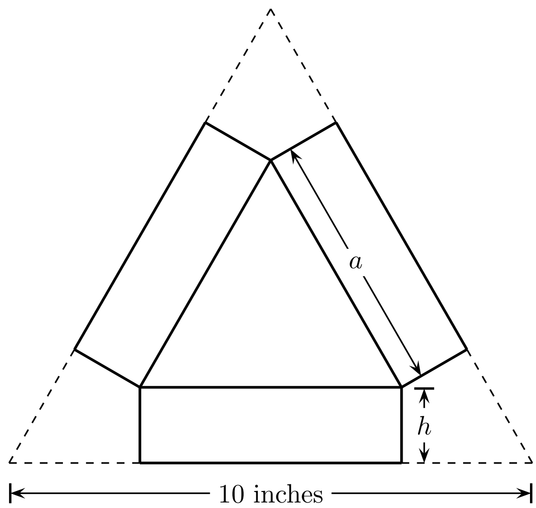
Find the following limits.

(a) $\displaystyle \lim_{x \to (\frac{\pi}{2})^+} \left[\left(x - \frac{\pi}{2} \right)^4 \sin(\tan x) \right]$.

(b) $\displaystyle \lim_{x \to 0} \left[\frac{(1 + 2x)^{\frac{1}{x}}}{e^2} \right]^{\frac{1}{x}}$.

Question 4 [8 marks]

An open-top gift box is to be made from a piece of cardboard that is an equilateral triangle with each side of length 10 inches. Three corners of the cardboard are cut in the way indicated by the figure below, followed by folding the tabs to form the gift box. The resulting box has a base which is an equilateral triangle with each side of length a, and its height is h. What are the values of a and h that will lead to the box with largest volume, and what is the largest volume?



Question 5 [14 marks]

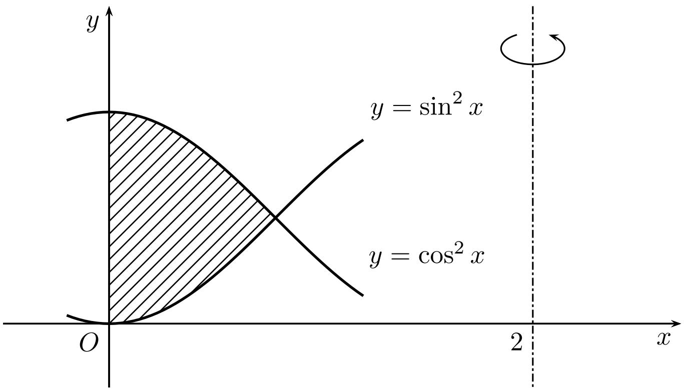
Evaluate the following integrals.

(a) $\displaystyle\int \frac{(\tan\theta + 4)\sec^2\theta}{\tan\theta(\tan^2\theta + 4)}\, d\theta.$

(b) $\displaystyle\int_1^e (x\ln x)^2\, dx.$

Question 6 [14 marks]

(a) Find the volume of the solid generated by revolving the region enclosed by the curves $y = \sin^2 x$, $y = \cos^2 x$ and the y-axis, shaded in the following figure, about the line $x = 2$.



(b) Find the area of the surface generated by revolving the curve $y = x^{1/3}$, $1 \le x \le 8$, about the y-axis.

Question 7 [15 marks]

(a) Solve the following differential equation with initial condition:

$$(x \ln x)\frac{dy}{dx} + y = 3x^3 \quad (x > 1), \qquad \text{where } y = 1 \text{ if } x = 2.$$

(b) The learning curve of a new worker on an assembly line is modeled by the differential equation

$$\frac{dq}{dt} = k(M - q),$$

where $q = q(t)$ is the number of units the worker completes during her t^{th} day on the assembly line, k and M are positive constants, and $q < M$ for all $t \ge 0$. In addition, it is assumed that $q(0) = 0$.

(i) A certain new worker on the assembly line completed 140 units during her first day and 200 units her second day. Based on the given differential equation, derive an expression for the number of units she completes during her t^{th} day.

(ii) How many units per day can the worker in part (i) eventually be expected to complete? Justify your answer.

Question 8 [15 marks]

(a) Let f be a function continuous on $[0,1]$ and twice differentiable on $(0,1)$. Suppose that

$$\int_0^1 f(x)\,dx = f(0) = f(1).$$

Prove that there exists a number $x_0 \in (0,1)$ such that $f''(x_0) = 0$.

(b) Let g be a function and c a number in its domain. Suppose that

$$g'(c) = g''(c) = g'''(c) = 0 \quad \text{and} \quad g^{(4)}(c) > 0.$$

Prove that g attains a local minimum at c.

[End of Paper]