NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2011-2012

MA1101R Linear Algebra I

November 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. Each question carries 25 marks.
3. Calculators may be used. However, you should lay out systematically the various steps in the calculations

Question 1

(a) [13 marks] Consider the following linear system.

$$\begin{cases} x + 4y + 5z + 2w = 0 \\ 2x + y + 3z - 3w = 0 \\ -x + 3y + 2z + 5w = 0 \end{cases}$$

(i) Use Gaussian Elimination or Gauss-Jordan Elimination to find a general solution for the linear system.

(ii) Determine the dimension of the solution space of the linear system.

(iii) Is $\{(3, -3, 1, 2), (-3, 0, 1, -1)\}$ a basis for the solution space of the linear system? Justify your answer.

(b) [12 marks] Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2s & 1 \\ s & 1 & 1 \\ s & 1 & 3s \end{pmatrix}$$

where s is a real number.

(i) Compute $\det(\boldsymbol{A})$ using elementary row operations.

(ii) Find all values of s such that the nullspace of $\boldsymbol{A}$ is the zero space. Justify your answer.

(iii) Find **one** value of s such that the nullspace of $\boldsymbol{A}$ is a straight line through the origin in $\mathbb{R}^3$. Justify your answer.

(iv) Are there values of s such that the column space of $\boldsymbol{A}$ is a straight line through the origin in $\mathbb{R}^3$? Justify your answer.

Question 2

(a) [13 marks] Let

$$\boldsymbol{B} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

where $\lambda \neq 0$.

(i) Briefly explain why $\boldsymbol{B}$ is invertible.

(ii) Find the smallest positive integer k such that $(\boldsymbol{B} - \lambda \boldsymbol{I})^k = \boldsymbol{0}$.

(iii) Find an expression for $\boldsymbol{B}^{-1}$ in terms of sums of powers of $\boldsymbol{B}$. (**Hint:** Use (ii))

(iv) Explain why for every $n \geq 1$,

$$V_n = \{\boldsymbol{v} \in \mathbb{R}^3 \mid (\boldsymbol{B} - \lambda \boldsymbol{I})^n \boldsymbol{v} = \boldsymbol{0}\}$$

is a subspace of $\mathbb{R}^3$.

(v) Find the smallest positive integer n such that $V_n = \mathbb{R}^3$.

(*Question 2 continues on next page...*)

(b) [6 marks] Let

$$\boldsymbol{C} = \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 1 & 1 & -1 & -3 & 3 \end{pmatrix}$$

and $\boldsymbol{R}$ be row equivalent to $\boldsymbol{C}$ such that

$$\boldsymbol{C} \xrightarrow{R_4 - R_1} \xrightarrow{R_2 \leftrightarrow R_4} \xrightarrow{R_4 + R_3} \boldsymbol{R}.$$

Find $\boldsymbol{R}$ and **three** elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ such that

$$\boldsymbol{C} = \boldsymbol{E_1 E_2 E_3 R}.$$

(c) [6 marks] Compute the projection of $(1,1,1,1)$ onto the row space of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Question 3

(a) [12 marks] Let $\boldsymbol{v_1} = (1,2,0,1)$, $\boldsymbol{v_2} = (1,0,1,1)$, $\boldsymbol{v_3} = (-1,-4,1,0)$.

(i) Write down a 3×4 matrix whose row space is $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$.

(ii) Find a basis for $V = \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ and determine $\dim(V)$ (**Hint:** Use (i))

(iii) Is it possible to find a vector $\boldsymbol{v_4}$ such that $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ is a basis for $\mathbb{R}^4$? If it is possible find one such vector $\boldsymbol{v_4}$.

(iv) Find a nonzero subspace W of $\mathbb{R}^4$ such that

$$\boldsymbol{w} \cdot \boldsymbol{v} = 0$$

for all $\boldsymbol{w} \in W$, $\boldsymbol{v} \in V$.

(b) [6 marks] Apply Gram-Schmidt Process to transform the basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

into an orthonormal basis for $\mathbb{R}^3$.

(c) [7 marks] Let T_1 and T_2 be two linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^1$. If $\ker(T_1) = \ker(T_2)$, show that there is a non zero constant a such that $T_1(\boldsymbol{v}) = aT_2(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

(*Question 4 is on the next page...*)

Question 4

(a) [13 marks] Let

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that $\boldsymbol{A}$ is symmetric.

(i) Find an orthogonal matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T$. (**Hint:** The eigenvalues of $\boldsymbol{A}$ are all integers.)

(ii) Prove or disprove the following statement:

If $\boldsymbol{P}$ orthogonally diagonalizes a symmetric matrix $\boldsymbol{A}$,

then $\boldsymbol{P}$ also orthogonally diagonalizes $k\boldsymbol{A}$ for any real number k.

(b) [7 marks] Suppose

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

is the standard matrix for a linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$.

(i) Determine m and n.

(ii) Determine nullity(T) and rank(T).

(iii) Find an orthonormal basis for the range of T.

(c) [5 marks] Let $\boldsymbol{C}$ be a $n \times n$ invertible matrix and $T = \{\boldsymbol{c_1}, \boldsymbol{c_2}, ..., \boldsymbol{c_n}\}$ where $\boldsymbol{c_i}$, $i = 1, ..., n$, is the i-th column of $\boldsymbol{C}$. If $S = \{\boldsymbol{e_1}, \boldsymbol{e_2}, ..., \boldsymbol{e_n}\}$ is the standard basis for $\mathbb{R}^n$, prove that $\boldsymbol{C}^{-1}$ is the transition matrix from S to T.

END OF PAPER