NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2011-2012

Ph.D. QUALIFYING EXAMINATION

Paper 2

ANALYSIS

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination contains a total of **TEN (10)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The maximum score for this examination is 100 points.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 points] For each of the following statements, prove it if it is true and provide a counterexample if it is false.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function so that the directional derivative of f in any direction exists at $(0,0)$. Then f is differentiable at $(0,0)$.

(b) Let U be a connected open set in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be an analytic function. If there exists $z_0 \in U$ such that $\operatorname{Re} f(z_0) \geq \operatorname{Re} f(z)$ for any $z \in U$, then f is constant on U.

Question 2 [10 points] Let X be a compact metric space. Show that there is a sequence of open sets $(U_n)_{n=1}^{\infty}$ in X such that for any $x_0 \in X$ and any closed set F in X not containing x_0, there exists n so that $x_0 \in U_n$ and $\overline{U_n} \cap F = \emptyset$.

Question 3 [10 points] Let f be a complex function that is analytic on an open set containing the closed ball $\{z \in \mathbb{C} : |z| \leq 1\}$. Assume that $f(0) \neq 0$ and that $f(z) \neq 0$ for any z with $|z| = 1$. Suppose that $(a_k)_{k=1}^{n}$ are the distinct zeros of f in $\{z \in \mathbb{C} : |z| < 1\}$, with respective multiplicities $(m_k)_{k=1}^{n}$. Show that

$$\sum_{k=1}^{n} \frac{m_k}{a_k^2} = \int_C \frac{f'(z)}{z f(z)}\, dz - \frac{f'(0)}{f(0)},$$

where C is the circle $\{z : |z| = 1\}$, traversed once in the counterclockwise direction.

Question 4 [10 points] Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue integrable function. Denote Lebesgue measure by λ. Show that the series

$$s_n = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \lambda\big(\{x : \frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}\}\big)$$

converges absolutely for each $n \in \mathbb{N}$, and that $\lim_{n\to\infty} s_n = \int_0^1 f\, d\lambda$.

Question 5 [10 points] For any $n \in \mathbb{N}$, the n-th Rademacher function $r_n : [0,1] \to \mathbb{R}$ is defined by

$$r_n(t) = \begin{cases} (-1)^{k+1} & \text{if } t \in [\frac{k-1}{2^n}, \frac{k}{2^n}),\ 1 \leq k \leq 2^n, \\ 0 & \text{if } t = 1. \end{cases}$$

Show that $\lim_{n\to\infty} \int_0^1 f r_n d\lambda = 0$ for any $f \in L^1[0,1]$. Here λ denotes Lebesgue measure.

Question 6 [10 points] Let $f, g : [0,1] \to \mathbb{R}$ be integrable functions with respect to Lebesgue measure λ. Assume that for any $a, b \in \mathbb{R}$

$$\lambda(\{t : f(t) \leq a\} \cap \{t : g(t) \leq b\}) = \lambda\{t : f(t) \leq a\} \cdot \lambda\{t : g(t) \leq b\}.$$

Show that fg is integrable on $[0,1]$ with respect to Lebesgue measure and that

$$\int_{[0,1]} fg \, d\lambda = \int_{[0,1]} f \, d\lambda \cdot \int_{[0,1]} g \, d\lambda.$$

Question 7 [10 points] Let $(f_k)_{k=1}^{\infty}$ be a sequence in $L^p(\mathbb{R})$, where $1 \leq p < \infty$. Suppose that $f_1 \leq f_2 \leq \cdots$ and $\sup_k \|f_k\|_p < \infty$. Show that $(f_k)_{k=1}^{\infty}$ converges in L^p norm.

Question 8 [10 points] Let f be a Lebesgue integrable function on $[0,1]$ and denote Lebesgue measure by λ. Suppose that $0 < \alpha < 1$. Show that for almost all $t \in [0,1]$, the function $F_t(x) = f(x)|x-t|^{-\alpha}$ is integrable on $[0,1]$. Define $g(t) = \int_0^1 F_t \, d\lambda$ where the integral exists and 0 otherwise. Show that $g \in L^1[0,1]$.

Question 9 [10 points] Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a,b) \to \mathbb{R}$ be a continuous function. Define F to be the set of all $x \in (a,b)$ such that $f'(x)$ exists (as a real number). For each $k \in \mathbb{N}$, and any $p, q, q' \in \mathbb{Q}$ with $a < q < q' < b$, define

$$H(k,p,q,q') = \{x \in (q,q') : |f(y) - f(x) - p(y-x)| \leq \frac{|y-x|}{k} \text{ for all } y \in (q,q')\}.$$

Express F in terms of the sets $H(k,p,q,q')$ and deduce that F is a Borel set.

Question 10 [10 points] Suppose that $1 \leq p < \infty$. Show that there is a linear bijection $T : L^p[0,1] \to L^p(\mathbb{R})$ such that $\int_{\mathbb{R}} |Tf|^p \, d\lambda = \int_0^1 |f|^p \, d\lambda$ for all $f \in L^p[0,1]$, where λ is Lebesgue measure.

END OF PAPER