Ph.D. Qualifying Examination 2011 August (Analysis)

(1) Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$. If $\{b_k\}$ is a permutation of $\{a_k\}$, show that [10]

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k.$$

(2) Let f be a finite measurable function on E with $|E| < \infty$. Show that given any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a compact set $F \subset E$ such that $|E \setminus F| < \varepsilon$ and $|f(x)| < N$ for $x \in F$. [8]

(3) If $f : \mathbb{C} \to \mathbb{C}$ is bounded analytic on a deleted neighborhood of a point z_0, show that z_0 is a removable singularity of f. [10]

(4) Let $f_n, f : \mathbb{R} \to \mathbb{R}$ be measurable functions. If $f_n \to f$ in measure and $f_n \geq 0$ for all n, show that [8]

$$\int f dx \leq \liminf_{n\to\infty} \int f_n dx.$$

(5) Let f be an entire function. Suppose for each $z \in \mathbb{C}$, there exists $n \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. Show that f must be a polynomial. [10]

(6) Let $1 \leq p < \infty$. Show that there exists $C > 0$ such that

$$\left(\int |\sum_{i=1}^{\infty} a_i \chi_{2Q_i}|^p dx\right)^{1/p} \leq C\left(\int |\sum_{i=1}^{\infty} |a_i| \chi_{Q_i}|^p dx\right)^{1/p}$$

for any sequence of real numbers a_i, cubes Q_i ($2Q_i$ is the cube with the same center as Q_i but twice its length). [10]

(7) (i) Let $R = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$. Let $u : R \to \mathbb{R}$ be a twice continuously differentiable function. Prove (without using divergence theorem) by using integration by parts or direct integration that [8]

$$\int_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dA = \int_{\partial R} \nabla u \cdot \vec{n} ds$$

where $\vec{n}$ is the outward unit normal vector.

(ii) Let $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$ on a domain Ω in $\mathbb{R}^2$. If ϕ is any C^∞ function that vanishes outside a compact subset of Ω, show that [4]

$$\int_\Omega \phi f dA = -\int_\Omega \nabla u \cdot \nabla \phi dA.$$

(8) Prove or disprove **Eight** (8) of the following statements. [32]

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, that is, for all $x \in \mathbb{R}$, there exist $L_x, \delta_x > 0$ such that

$$|f(x) - f(y)| \leq L_x|x - y| \text{ if } |y - x| < \delta_x.$$

Then $f(E)$ is measurable whenever E is measurable.

(b) Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a continuous function. If Ω is simply connected and $\int_\gamma f(z)dz = 0$ for any closed contour γ in Ω, then f is analytic on Ω.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is measurable and nonnegative, then $f = \sum_{k=1}^{\infty} \frac{1}{k}\chi_{A_k}$ where A_k are measurable sets.

(d) Let $\Omega = \{z \in \mathbb{C} : 1 < |z| < 3\}$ and f is a bounded analytic function on Ω. If there exists $z_0 \in \mathbb{C}$, $|z_0| = 2$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, then $|f(z)| = |f(z_0)|$ for all $z \in \Omega$.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable a.e., then its derivative is measurable.

(f) If $\{a_k\}$ is a monotone sequence of real numbers that converges to 0, then $\sum_{k=1}^{\infty} a_k \cos kx$ converges for almost all $x \in \mathbb{R}$.

(g) If a function is continuous a.e., then it is measurable.

(h) Let $f : [a, b] \to \mathbb{R}$. If there exists $M > 0$ such that for all $a < x_0 < x_1 \cdots < x_n < b$,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M,$$

then f is a function of bounded variation on $[a, b]$.

(i) Let $\{f_n\}$, $\{g_n\}$ be two sequences of measurable functions on $\mathbb{R}$ such that $f_n \to f$ a.e. and $|f_n| \leq g_n$ a.e.. If there exists a measurable function g such that

$$\lim_{n\to\infty} \int g_n dx = \int g dx,$$

then

$$\lim_{n\to\infty} \int f_n dx = \int f dx.$$

(j) Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$. If for each $x, y \in \mathbb{R} \setminus \{0\}$, there exists L such that

$$\lim_{t\to 0} |f(xt, yt) - f(-xt, -yt) - Lt|/t = 0,$$

then f is differentiable at $(0, 0)$.

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