NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2010-2011

QF3101 Investment Instruments: Theory and Computation

April/May 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and **ONE** (1) appendix and comprises **SEVEN (7)** printed pages (including this cover page).
2. Answer **ALL** questions. The mark for each question is indicated at the beginning of the question.
3. Start your answer to each question on a new page.
4. This is a closed book examination. Use of help sheets is **not** allowed.
5. You may use a non-programmable calculator. However, you should lay out systematically the various steps in the calculations.
6. Express all numerical answers up to **4 decimal places**.

Question 1 [*20 marks*]

(a) Consider the two-factor model for asset i, $i = 1, 2$,

$$r_i = a_i + b_{1i} f_1 + b_{2i} f_2 + e_i$$

where $\mathrm{E}(e_i) = 0$, $\mathrm{cov}(e_i, f_1) = \mathrm{cov}(e_i, f_2) = 0$, and $\mathrm{cov}(e_i, e_j) = 0$ for $i \neq j$.

(i) Use the definition $\mathrm{cov}(r_i, r_j) = \mathrm{E}[(r_i - \overline{r}_i)(r_j - \overline{r}_j)]$ to derive an expression for $\mathrm{cov}(r_i, r_j)$, $i \neq j$, in terms of the variances and covariance of the two factors.

(ii) The following estimates of the covariances have been obtained:

	Covariance with			
	Asset 1	Asset 2	Factor 1	Factor 2
Factor 1	0.25	0.05	0.92	0.12
Factor 2	0.08	0.21	0.12	0.72

Determine the factor loadings for both assets 1 and 2.

(b) Suppose that the market can be described by the following three sources of systematic risk with associated factor prices:

Factor	Factor Price
Industrial Production (I)	8%
Interest Rates (R)	2%
Consumer confidence (C)	5%

The returns of stocks 1 and 2 are generated according to the following equation:

$$\begin{aligned} r_1 &= 15\% + 1.0 f_I + 0.5 f_R + 0.75 f_C + e_1 \\ r_2 &= 8\% + 0.5 f_I + 1.2 f_R + 1.05 f_C + e_2 \end{aligned}$$

where e_1 and e_2 are random error terms; and f_I, f_R and f_C measure the deviations from some anticipated values for the three factors. As such $\mathrm{E}(f_I) = \mathrm{E}(f_R) = \mathrm{E}(f_C) = \mathrm{E}(e_1) = \mathrm{E}(e_2) = 0$. A portfolio is constructed from these two stocks with the weight for stock 1 being $w_1 = 1/8$.

(i) If the risk-free rate is 3%, find the equilibrium rate of return on this portfolio using the arbitrage pricing theory.

(ii) Is the portfolio over-priced or underpriced? Explain.

Question 2 [*20 marks*]

(a) On day X, a bank wishes to lock in the purchase price of a 2-year 4% Treasury note of $50,000 par value to be issued by the Treasury 6 months later. The bank decides that a forward contract is to be used to achieve this objective. The current 6-month, 12-month, 18-month, 24-month, 30-month spot rates are 1.2%, 1.8%, 2.2%, 2.8% and 3.4% (annualized, semi-annual compounding) respectively.

(i) Determine the key terms of the forward contract which the bank could use to achieve its objective. Also determine the forward dollar price for this contract.

(ii) Suppose six months later, immediately before the delivery of the Treasury note, the 6-month, 12-month, 18-month, 24-month, 30-month spot rates are 1.25%, 1.9%, 2.3%, 2.9% and 3.5% (annualized, semi-annual compounding) respectively. What is the value of the bank's forward position?

(b) Time now is 0 and a forward contract on a dividend-paying stock expires at time T. The current stock price is S_0 and the dividend of $\$D$ is paid at time t, $0 < t < T$. The appropriate forward price for this contract is given by

$$F_0 = \frac{S_0}{d(0,T)} - \frac{D}{d(t,T)}$$

where $d(k,T)$ is the discount factor from time k to time T. Suppose an arbitrageur observes that $F_0 < \dfrac{S_0}{d(0,T)} - \dfrac{D}{d(t,T)}$.

(i) Give an arbitrage strategy that he/she could undertake for a profit.

(ii) Explain how such arbitragers' actions would eventually help to restore the equality.

Question 3 [*20 marks*]

(a) On day Y, a 2-year interest rate swap is to be configured on half-yearly in-arrear exchanges of a fixed rate payment for a floating level payment linked to a 6-month rate on a notional principal of $100,000. The current 6-month, 12-month, 18-month and 24-month spot rates are 2.2%, 2.8%, 3.6% and 4.4% (annualized, semi-annual compounding) respectively.

(i) Determine the swap rate of the 2-year interest rate swap.

(ii) Determine the yield-to-maturity of a 2-year bond (paying semi-annual coupons) trading at par on day Y.

(iii) Suppose one year later, immediately after the 2nd swap exchange for the interest rate swap, the swap value is negative for the fixed rate payer. What can you say about the dollar price of a newly issued one-year bond paying semi-annual coupons at the same annual rate as the swap rate of the 2-year swap?

(b) Company X wishes to borrow at a fixed rate of interest for 5 years while Company Y wishes to borrow at a floating rate of interest for the same 5-year period. They have been quoted the following rates per annum:

	Fixed Rate	**Floating Rate**
Company X	3.5%	LIBOR+2.15%
Company Y	3.2%	LIBOR+2.0%

Design a swap arrangement, with a bank acting as intermediary, that will allow the parties involved to share the rate savings equally.

Question 4 [*20 marks*]

(a) Suppose the dollar price of a treasury bill (TB1) with 60 days to maturity is $98.22, and that of another treasury bill (TB2) with 150 days to maturity is $97.13.

(i) Determine the discount yield and investment rate for TB1. Assume 365 days in the year concerned.

(ii) If the futures price (index basis) of a treasury bill futures which expires 60 days from now is 98.12, determine the implied repo rate for 60 days from now.

(iii) Is there an arbitrage opportunity? If so, briefly describe the arbitrage strategy.

(b) Suppose today is 18 June 2011, and the following table gives a summary of information on six Eurodollar futures with settlement prices recorded at the end of the day.

Contract	Settlement Price	Contract Expiry Date	#Days from 18-June to Expiry Date
Jun 11	99.685	18-Jun-2011	0
Sep 11	99.575	17-Sep-2011	91
Dec 11	99.400	17-Dec-2011	182
Mar 12	99.125	18-Mar-2012	274
Jun 12	99.755	17-Jun-2012	365
Sep 12	98.360	16-Sep-2012	456

(i) Determine the fair fixed rate for a 6×12 forward rate agreement initiated today.

(ii) A 9×15 forward rate agreement on $100,000 notional principal was initiated six months ago with the fixed rate 0.8125%, determine the current value of a short position on this forward rate agreement.

Question 5 [*20 marks*]

(a) Consider a portfolio consisting of a $300,000 investment in asset 1, a $200,000 investment in asset 2 and a $500,000 investment in asset 3. Assume that the assets' returns are normally distributed, the daily volatilities of the assets are 1.8%, 2.1% and 1.2% respectively, and that the coefficients of correlation between their returns are $\rho_{12} = \rho_{13} = \rho_{23} = 0.3$. For a risk horizon of 5 days, a year of 250 trading days and a confidence level of 95%, determine the undiversified VaR and the diversified VaR of the portfolio.

(b) A bond trading book holds a position on a zero coupon bond with a face value of $1,000,000. This zero will mature in 0.75 years.

(i) Given the following information, use cash flow mapping to map this position to the adjacent standard time vertices of 0.5 and 1 year.

Standard	Yield	Price Volatility	Correlation Matrix, R	
Maturity	(%)	(daily %)	0.5 year	1 year
0.5 year	2.18	0.12	1	0.97
1 year	2.09	0.18	0.97	1

(ii) Determine the 1-day 99% VaR of this zero-coupon bond position.

END OF PAPER

Appendix: QF3101 Formula Sheet (Two pages)

For asset i, we have

- Capital Asset Pricing Model (CAPM): $\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f)$, where $\beta_i = \sigma_{iM}/\sigma_M^2$.
- Systematic and Firm-Specific Risk: $\sigma_i{}^2 = \beta_i^2\sigma_M^2 + \text{var}(\epsilon_i)$

Single Factor Models For asset i, we have $r_i = a_i + b_i f + e_i, i = 1, 2, \ldots, n$, and

$$\bar{r}_i = a_i + b_i\bar{f}, \qquad \sigma_i^2 = b_i^2\sigma_f^2 + \sigma_{e_i}^2$$
$$\sigma_{ij} = b_i b_j \sigma_f^2, \quad i \neq j, \qquad b_i = \text{cov}(r_i, f)/\sigma_f^2$$
$$R^2 = (\text{var}(r_i) - \text{var}(e_i))/\text{var}(r_i).$$

For a portfolio of assets following single-factor models: $\sigma_p^2 = b^2\sigma_f^2 + \sigma_e^2$ where $b = \sum_{i=1}^n w_i b_i$, $e = \sum_{i=1}^n w_i e_i$ and $\sigma_e^2 = \sum_{i=1}^n w_i^2 \sigma_{e_i}^2$.

Two-Factor Models For asset i, we have $r_i = a_i + b_{1i}f_1 + b_{2i}f_2 + e_i, i = 1, 2, \ldots, n$, and

$$\begin{aligned}
\overline{r}_i &= a_i + b_{1i}\overline{f}_1 + b_{2i}\overline{f}_2, \\
\text{cov}(r_i, r_j) &= b_{1i}b_{1j}\text{var}(f_1) + (b_{1i}b_{2j} + b_{2i}b_{1j})\text{cov}(f_1, f_2) + b_{2i}b_{2j}\text{var}(f_2) + \text{cov}(e_i, e_j) \\
\text{cov}(r_i, f_1) &= b_{1i}\,\text{var}(f_1) + b_{2i}\,\text{cov}(f_1, f_2) \\
\text{cov}(r_i, f_2) &= b_{1i}\,\text{cov}(f_1, f_2) + b_{2i}\,\text{var}(f_2).
\end{aligned}$$

Arbitrage Pricing Theory For asset i on m-factor model: $r_i = a_i + \sum_{j=1}^m b_{ji}f_j + e_i$, APT implies that
$\overline{r}_i = \lambda_0 + \sum_{j=1}^m b_{ji}\lambda_j$, where λ_0 is the risk-free rate, and λ_j is the factor price for factor j, $j = 1, \ldots, n$.

Forward Price Formulae

- $F_0 = S_0/d(0, T)$,
- $F_0 = (S_0 - I)/d(0, T)$,
- $F_0 = S_0 e^{(r-q)T}$
- $F_0 = \dfrac{S_0}{d(0, M)} + \displaystyle\sum_{k=0}^{M-1} \frac{c(k)}{d(k, M)}$
- $F_0 = \dfrac{S_0}{d(0, M)} + \displaystyle\sum_{k=0}^{M-1} \frac{c(k)}{d(k, M)} - \sum_{k=0}^{M-1} \frac{y}{d(k, M)}$

Interest Rate Swap The initial value of an interest rate swap with maturity T to the company receiving floating and paying fixed on notional principal N is

$$V = \left(1 - d(0, T) - r\sum_{i=1}^M d(0, t_i)\right) N.$$

The value of swap with time t_1 to the next exchange for the company receiving fixed and paying floating is

$$V = Nd(0, T) + \sum_{i=1}^n k\ d(0, t_i) - (N + f_1)d(0, t_1).$$

Currency Swap The value of swap for the company paying currency 1 and receiving currency 2 is $V = SP_2 - P_1$, where S is the spot exchange rate, and P_i, $i = 1, 2$, are the values of the bonds in currencies 1 and 2 respectively.

Hedge Basis $b_i = S_i - F_i$, Effective price for hedging $F_1 + b_2$,
Minimum variance hedge ratio $\beta^* = \dfrac{\text{cov}(S_T, F_T)}{\text{var}(F_T)}$.

Changing beta of a Portfolio No. of index contracts$= (\beta_1 - \beta_2)\dfrac{S}{m\,I_0}$.

T-bill discount yield$= \dfrac{100 - P}{100} \times \dfrac{360}{t}$, Investment rate (less than half year) $i = \dfrac{100 - P}{P} \times \dfrac{y}{t}$.

For more than one half-year to maturity, solve $(t/(2y) - 0.25)i^2 + (t/y)i + (P - 100)/P = 0$.

Full price of a bond$= \displaystyle\sum_{i=0}^{n-1} \frac{C}{\left(1 + \frac{r}{m}\right)^{i+w}} + \frac{F}{\left(1 + \frac{r}{m}\right)^{w+n-1}}$, accrued interest$= (1 - w)C$.

Add-on yield $= \dfrac{\text{Interest}}{\text{Principal}} \times \dfrac{360}{\text{Days to maturity}}$.

Eurodollar Futures Price Index Price$= 100-$(annualized add-on yield for the 3-month period).

FRA payoff to buyer $(y - X) \times N \times \dfrac{m}{360} \times \dfrac{1}{1 + y(m/360)}$.

Forward rate for $[T, T + m]$ $f_L(T, T + m) = \left(\dfrac{d_L(T)}{d_L(T + m)} - 1\right) \Big/ (m/360)$.

PV of FRA $[f_L(T, T + m) - X] \times (m/360) \times d_L(T + m)$.

Currency Forward Forward Price $F_0 = x_0\, d(r_f, T)/d(r_l, T)$, r_f and r_l are foreign and local interest rates respectively.

Value-at-Risk
For single position, relative VaR: $\text{VaR}_r = \alpha\sigma W_0$, absolute VaR: $\text{VaR}_a = (\alpha\sigma - \mu)W_0$.

$\alpha = 1.65$ for 95% confidence level, and $\alpha = 2.33$ for 99% confidence level.

Square root of time rule N-day VaR$=$ 1-day VaR$\times\sqrt{N}$.

Delta-Normal method Portfolio VaR is $\text{VaR}_p = \alpha\sqrt{\mathbf{x}'\mathbf{\Sigma}\mathbf{x}}$.

Cash Flow Mapping For mapping zero at t to zeros at standard maturities p and q with $p < t < q$.
Solve quadratic for alpha: $(\sigma_p^2 + \sigma_q^2 - 2\rho_{p,q}\sigma_p\sigma_q)\alpha^2 + (2\rho_{p,q}\sigma_p\sigma_q - 2\sigma_q^2)\alpha + (\sigma_q^2 - \sigma_t^2) = 0$.