NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2010-2011

MA3501 Introductory Mathematics

April/May 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in the examination paper. Marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

4. One A4 handwritten double-sided helpsheet is allowed.

Answer all the questions.

Question 1 [20 marks]

(a) Let X be the number of oil tankers arriving each week at a certain port. Suppose that X has a Poisson distribution with mean λ. Prove that

$$\frac{P(X = r+1)}{P(X = r)} = \frac{\lambda}{r+1}$$

for $r = 0, 1, 2, 3, \cdots$.

If $\lambda = 15$, find the value (or values) of r for which $P(X = r)$ is the greatest.

(b) A commonly prescribed drug on the market for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension showed that 70 of them had their tension relieved. Test, at the 5% significant level, whether the new drug is superior to the one commonly prescribed.

(c) (i) Solve, by the method of characteristics, the following first order partial differential equation.

$$u_t(x,t) + u_x(x,t) + u(x,t) = 0, \; u(x,0) = f(x).$$

(ii) Given that $\int_{-\infty}^{\infty} f(x)\,dx = 1$, express $\int_{-\infty}^{\infty} u(x,t)\,dx$ in terms of t.

(d) Find a particular solution of the system

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2t \end{pmatrix}.$$

Question 2 [20 marks]

(a) Find the steady state solution of the boundary value problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + \sin x, && 0 < x < \pi,\ t > 0, \\ u(0,t) &= 500, && t > 0, \\ u(\pi,t) &= 100, && t > 0. \end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + \sin x, && 0 < x < \pi,\ t > 0, \\ u(0,t) &= 500, && t > 0, \\ u(\pi,t) &= 100, && t > 0, \\ u(x,0) &= \sin x + 500, && 0 < x < \pi. \end{aligned}$$

(c) Solve the boundary value problem

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0, && -\infty < x < \infty,\ y > 0, \\ u(x,0) &= \cos x, && -\infty < x < \infty. \end{aligned}$$

Question 3 [20 marks]

(a) Find a real number α and two functions $f(x)$, $g(x)$ defined on $(-\infty, \infty)$ such that $u(x,t) = f(x+\alpha t) + g(x - \alpha t)$ is the solution to the initial value problem

$$\begin{aligned} u_{tt}(x,t) &= 9u_{xx}(x,t), && -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \sin x, && -\infty < x < \infty,\ t > 0, \\ u_t(x,0) &= \sin x(1+\cos x), && -\infty < x < \infty. \end{aligned}$$

(b) Solve the boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t), && 0 < x < \infty,\ t > 0, \\ u(x,0) &= 0, && 0 < x < \infty, \\ u_t(x,0) &= 0, && 0 < x < \infty, \\ u(0,t) &= \sin t, && t > 0, \\ |u(x,t)| &< 100, && 0 < x < \infty,\ t > 0. \end{aligned}$$

Question 4 [20 marks]

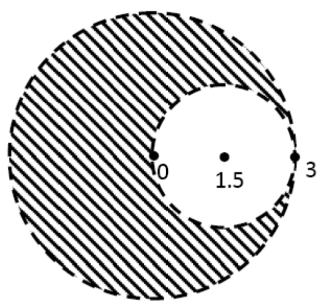
(a) Evaluate

$$\int_\gamma \left[ze^{3/z} + \frac{\cos z}{z^2(z-\pi)^3}\right]\, dz,$$

where γ is the anticlockwise oriented circle with centre $(0,0)$ and radius 3.

(b) Find the image of the disk $|z| < 3$ under the mapping $f(z) = \dfrac{z+3}{z-3}$.

(c) Let $D = \{(x,y) : x^2+y^2 < 9\} \setminus \{(x,y) : (x-\frac{3}{2})^2 + y^2 \leq \frac{9}{4}\}$, that is, D is the disk $x^2+y^2<9$ in which the disc $(x-\frac{3}{2})^2+y^2 \leq \frac{9}{4}$ is cut out, as shown in the shaded region below.



Solve the Laplace equation

$$\begin{array}{lll} \phi_{xx} + \phi_{yy} = 0 & \text{on} & D, \\ \phi(x,y) = 1 & \text{on} & x^2+y^2 = 9, \\ \phi(x,y) = 0 & \text{on} & (x-\frac{3}{2})^2 + y^2 = \frac{9}{4}. \end{array}$$

END OF PAPER