NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2010-2011

MA3227 Numerical Analysis II

May 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**. Each question in Section A carries 20 marks.
3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. This is a closed book exam.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **ALL** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [20 marks]

(a) Prove that if A is invertible, then $\|Ax\| \geq \|x\|\|A^{-1}\|^{-1}$ where $\|A^{-1}\|$ is the matrix norm induced by the vector norm $\|x\|$.

(b) Prove that if A is invertible and if λ is an eigenvalue of A, then $\|A^{-1}\|^{-1} \leq |\lambda| \leq \|A\|$. Here the matrix norm is induced by any vector norm.

Question 2 [20 marks]

Let $f(x) = \frac{1}{2}x^\top Ax - x^\top b$ where $x, b \in \mathbb{R}^{n\times 1}$ and $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. The following is the steepest decent algorithm to find $x_* = \text{argmin}_x f(x)$ which satisfies $Ax_* = b$.

1. Initialize with x_0
2. For $k = 1, 2, ...$ until some stopping criteria is satisfied
   - $r_{k-1} = b - Ax_{k-1}$.
   - Find α_k so that $\alpha_k = \text{argmin}_\alpha f(x_{k-1} + \alpha\, r_{k-1})$.
   - $x_k = x_{k-1} + \alpha_k r_{k-1}$.

(a) Derive an explicit formula for α_k.

(b) Prove that $r_{k-1}^\top r_k = 0$.

Question 3 [20 marks]

Show that the Newton's method for the function $f(x) = x^r - a$, $x > 0$, where $r > 1$ and $a > 0$, converges globally to $b = a^{\frac{1}{r}} > 0$ as long as the initial guess $x_0 \geq b$. [Hint: You can use the fact that a bounded monotone sequence converge to a finite number. The r is not necessarily an integer. But you will get half credit if you present a correct proof for some specific r, say, $r = 2$.]

SECTION B

Answer not more than **TWO** *questions in this section. Each question in this section carries 20 marks.*

Question 4 [20 marks]

Suppose you want to solve $Ax = b$ by the Gauss-Seidel iteration.

(a) Write down the equation satisfied by the error $e_k = x_k - x$ where x is the exact solution and x_k is the solution at the kth iteration. [5 marks]

(b) Let $A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$. Determine the necessary and sufficient condition for α so that the Gauss-Seidel iteration converges to the exact solution. [15 marks]

Question 5 [20 marks]

Consider the integration $I = \int_0^1 f(x)dx$. Given any probability density function (pdf) $g(x)$ with $\int_0^1 g(x)dx = 1$, you can rewrite I as $I = \int_0^1 \frac{f(x)}{g(x)} g(x)dx$. Let X be a random variable with pdf g and Let $U \sim U(0,1)$.

(a) Express $\text{var}_U(f(U))$ and $\text{var}_X\left(\frac{f(X)}{g(X)}\right)$ as integrals on $[0,1]$.

(b) Show that the g that minimizes $\text{var}_X\left(\frac{f(X)}{g(X)}\right)$ is

$$g^*(x) = \frac{|f(x)|}{\int_0^1 |f(x)|dx}.$$

[Hint: You may use the fact that $\left(\int_0^1 h(x)k(x)dx\right)^2 \leq \int_0^1 h^2 dx \int_0^1 k^2 dx$ and the "$\leq$" becomes "$=$" when $h(x) = ck(x)$ $\forall x$ with some constant c. What do you obtain if you take $h(x) = \frac{|f(x)|}{\sqrt{g(x)}}$ and $k(x) = \sqrt{g(x)}$?]

Question 6 [20 marks]

Let $A : [0,\infty) \to [0,1]$ be any function such that $A(z) = zA(1/z)$ for all $z \in [0,\infty)$. Let f be any probability density function. The following is the Metropolis-Hastings algorithm to generate a Markov chain with stationary distribution f:

Select a state x^0 as the initial state of the chain. Then:

for $t = 0, 1, 2, ...$

- Propose a random "perturbation" of the current state x^t so as to generate a new state y. Mathematically, $x^t \to y$ can be viewed as one step of a Markov chain with transition probability function $P_{x^t,y}$.
- Compute $z = \frac{P_{y,x^t}}{P_{x^t,y}} \frac{f(y)}{f(x^t)}$ and $h = A(z)$.
- Generate $u \sim U(0,1)$. If $u \leq h$, then $x^{t+1} = y$. Otherwise, $x^{t+1} = x^t$.

end

(a) Show both $A(z) = \min(1, z)$ and $A(z) = \frac{z}{1+z}$ satisfy the requirements $A(z) = zA(1/z)$ and $0 \leq A(z) \leq 1$ for all $z \in [0,\infty)$. [5 marks]

(b) Derive the transition probability function $Q_{x,y} = P(x^{t+1} = y | x^t = x)$ when $x \neq y$ for the Metropolis-Hastings chain x^t. [5 marks]

(c) Show that $f(x)Q_{x,y} = f(y)Q_{y,x}$ for any states x and y. [5 marks]

(d) Rename all the possible states by $\{1, 2, 3, ..., N\}$. Show that $\vec{f}Q = \vec{f}$ where $\vec{f} = (f(1), f(2), ..., f(N))$ and $Q = (Q_{i,j}) \in \mathbb{R}^{N\times N}$. [5 marks]

END OF PAPER