NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2010-2011

MA2214 Combinatorial Analysis

April 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Find the number of ways to distribute 12 balls to 3 boxes in each of the following cases:

   (i) Balls are identical, boxes are distinct and each box must contain at least one ball;

   (ii) Balls are identical, boxes are identical and each box must contain at least one ball;

   (iii) Balls are distinct, boxes are identical and each box must contain at least one ball.

(b) Find the number of ways to distribute 5 balls to 8 boxes in each of the following cases:

   (i) Balls are distinct, boxes are distinct and each box can contain at most one ball;

   (ii) Balls are identical, boxes are distinct and each box can contain at most one ball;

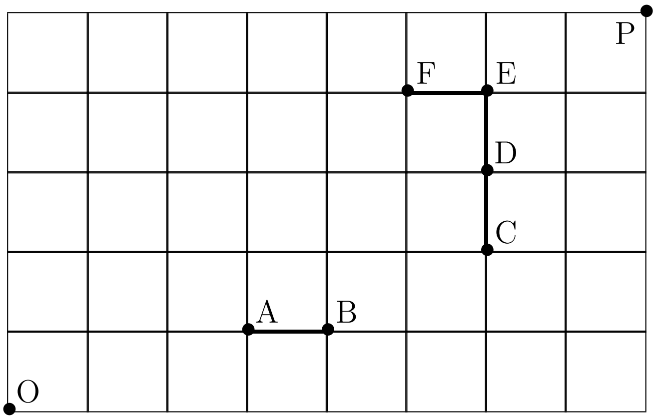
   (iii) Balls are identical, boxes are identical and each box can contain at most one ball.

Indicate clearly which answer corresponds to which part and leave your final answers as integers.

Question 2 [10 marks]

A palindrome is a number that remains the same when read either forwards or backwards, for example, 12321 or 954459. Find the total number of proper positive integers less than 1,000,000 that are palindromes.

Question 3 [20 marks]
Consider the following street network.



Find the number of shortest paths (NE lattice paths) from O to P that

(i) pass through FE;

(ii) that pass through exactly three of the four sections AB, CD, DE, FE;

(iii) that pass through exactly two of the four sections AB, CD, DE, FE;

(iv) that pass through exactly one of the four sections AB, CD, DE, FE;

(v) that does not pass through any of the four sections AB, CD, DE, FE.

Leave your answers as integers.

Question 4 [5 marks]

Prove that the number of partitions of n into parts each of which appears at most d times is equal to the number of partitions of n into parts whose sizes are not divisible by $d + 1$.

Question 5 [5 marks]
Let a_n be the number of ternary sequences in the letters X, Y and Z, with an even number of Y and at least one Z. Find a_n.

Question 6 [10 marks]

Let S be the set of proper three digit positive integers less than 1000.

(i) What is the minimum number of integers that must be chosen from S to ensure that there are at least two integers which start with the same digit?

(ii) What is the minimum number of integers that must be chosen from S to ensure that there are at least two integers which have at least one digit in common? (For example, 125 and 537 or 981 and 483).

Justify your answer.

Question 7 [10 marks]
Solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = \binom{n+4}{4}$ for $n \geq 2$, $a_0 = 0$ and $a_1 = 5$.

Question 8 [10 marks]
Let $g(x)$ be a polynomial of degree 7. Suppose that $g(k) = 3^k$ for $k = 0, 1, \ldots 7$, find $g(8)$.

Question 9 [10 marks]
How many arrangements of the letters $a, a, a, a, b, b, c, c, d, d$ are there such that no two consecutive letters are the same?

END OF PAPER