NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION, AY 2010/2011

MA2101 Linear Algebra II

May 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions.
3. Marks for each question are indicated at the beginning of the question. The marks for questions are not necessarily the same.
4. Calculators may be used. However, various steps in the calculations should be systematically laid out.

Question 1 [15 marks]

Let V and W be vector spaces and let

$$\varphi : V \to W$$

be a *surjective* linear transformation.

(a) Let V_1 be a subspace of V. Show that the image

$$\varphi(V_1) = \{\varphi(\mathbf{v}) \mid \mathbf{v} \in V_1\}$$

of V_1 is a subspace of W.

(b) Let S be a non-empty *subset* of W.

(bi) Show that the pre-image

$$\varphi^{-1}(S) = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) \in S\}$$

of S is a subspace of V if and only if S is a subspace of W.

(bii) Is (bi) still true if φ is not surjective? Justify your answer.

(c) Fix a vector $\mathbf{w}_0 \in W$. Suppose that the pre-image

$$\varphi^{-1}(\mathbf{w}_0) := \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) = \mathbf{w}_0\}$$

is a subspace of V. Show that $\varphi^{-1}(\mathbf{w}_0)$ equals the kernel

$$\mathrm{Ker}(\varphi) = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) = \mathbf{0}\}.$$

Question 2 [10 marks]

Let U and W be subspaces of a vector space V over a field F. Suppose that

$$V = U + W.$$

Define the following map between U and the quotient space V/W:

$$\begin{aligned} f : U &\to V/W = \{\overline{\mathbf{v}} = \mathbf{v} + W \mid \mathbf{v} \in V\}, \\ \mathbf{u} &\mapsto f(\mathbf{u}) := \overline{\mathbf{u}}. \end{aligned}$$

(i) Show that f is a linear transformation.

(ii) Show that f is surjective.

(iii) Show that f is an isomorphism if and only if $V = U \oplus W$, i.e., V is a direct sum of U and W.

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Question 3 [10 marks]

Let $A \in M_n(\mathbf{C})$ be a complex matrix such that $A^n = A$.

(a) Is A diagonalizable over $\mathbf{C}$? Justify your answer.

(b) Find all possible Jordan canonical forms of A. Justify your answers.

Question 4 [10 marks]

Let $B \in M_n(\mathbf{C})$ be an invertible complex matrix. Prove that there are matrices B_s, B_u in $M_n(\mathbf{C})$ such that the following three conditions are satisfied (justification should be clearly given in deducing (i) - (iii)):

(i) $B = B_s\, B_u$;

(ii) B_s is diagonalizable over $\mathbf{C}$, and 1 is the only eigenvalue of B_u; and

(iii) $B_s\, B_u = B_u\, B_s$.

Question 5 [10 marks]

Let $(V, \langle , \rangle)$ be a complex inner product space. Let W be a subspace of V and let B_1 be an orthonormal basis of W. Extend B_1 to a basis B' of V and apply the Gram-Schmidt process to B' to get an orthonormal basis $B = (B_1, B_2)$ of V. Define

$$W^{\perp} := \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall\, \mathbf{w} \in W\}.$$

(i) Show that $W^{\perp} = \mathrm{Span}(B_2)$.

(ii) Show that $V = W \oplus W^{\perp}$.

(iii) Suppose that $T : V \to V$ is a linear operator such that W is T^*-invariant, where T^* is the adjoint of T. Show that $W^{\perp}$ is T-invariant.

Question 6 [15 marks]

Let V be an n-dimensional vector space over a field F. Let U be the transition matrix from a basis B_2 of V to another basis B_1 of V so that $B_2 = B_1 U$.

(a) Show that U is an invertible matrix in $M_n(F)$.

(b) Suppose further that $F = \mathbf{C}$ and $(V, \langle , \rangle)$ is a complex inner product space and B_1 and B_2 are two orthonormal bases of V. Show that the transition matrix U in (a) is a unitary matrix, i.e., $U^* U = I_n$, where $U^* := \overline{(U^t)}$ is the adjoint, i.e., conjugate transpose of U.

Question 7 [15 marks]

(a) Let $T : V \to V$ be a linear operator on a finite-dimensional complex vector space V. Suppose that λ is a zero of the minimal polynomial $m_T(x)$ of multiplicity m_1, i.e., $(x - \lambda)^{m_1} \,|\, m_T(x)$ and $(x - \lambda)^{m_1+1} \not| \, m_T(x)$. Show that

$$\mathrm{Ker}(T - \lambda I_V)^{m_1} = \mathrm{Ker}(T - \lambda I_V)^r$$

for all $r \geq m_1$.

Note. You may assume, without proof, the fact that if $f_1(x)$ and $f_2(x)$ are complex polynomials with no common zero then $f_1(x)u(x) + f_2(x)v(x) = 1$ for some complex polynomials $u(x)$ and $v(x)$.

(b) Let $A \in M_n(\mathbf{C})$ be a complex matrix. Suppose that

$$A = A_s + A_n$$

is an (additive) Jordan decomposition such that A_s is diagonalizable, A_n is nilpotent,

$$A_s\, A_n = A_n\, A_s$$

and

$$A_s = f(A), \quad A_n = g(A)$$

for some polynomials $f(x)$, $g(x)$. Show that if

$$A = A'_s + A'_n$$

is another decomposition such that A'_s is diagonalizable, A'_n is nilpotent and

$$A'_s\, A'_n = A'_n\, A'_s,$$

then

$$A'_s = A_s, \quad A'_n = A_n.$$

Question 8 [15 marks]

For complex numbers $x_1, \ldots, x_n$; $y_1, \ldots, y_n$, denote by

$$X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad Y = (y_1, \ldots, y_n),$$

$$X^t = (x_1, \ldots, x_n), \quad \overline{Y} = (\overline{y_1}, \ldots, \overline{y_n}).$$

(a) Let $D \in M_n(\mathbf{C})$ be a positive definite complex matrix. Let W be an n-dimensional complex vector space with a basis $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. Show that the function below

$$W \times W \to \mathbf{C},$$

$$(\sum_{i=1}^{n} x_i \mathbf{w}_i, \sum_{j=1}^{n} y_j \mathbf{w}_j) \mapsto X^t \, D \, \overline{Y}$$

defines a complex inner product so that $(W, \langle, \rangle)$ is a complex inner product space, i.e., to verify the three conditions in the definition of an inner product space.

(b) Let $(V, \langle, \rangle)$ be an n-dimensional (*not necessarily standard*) complex inner product space. Let $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be a (*not necessarily orthonormal*) basis of V. Set

$$a_{ij} := \langle \mathbf{v}_i, \mathbf{v}_j \rangle, \quad A := (a_{ij}) \in M_n(\mathbf{C}).$$

(bi) Show that

$$\langle \sum_{i=1}^{n} x_i \mathbf{v}_i, \sum_{j=1}^{n} y_j \mathbf{v}_j \rangle := X^t \, A \, \overline{Y}.$$

(bii) Show that the matrix A is self-adjoint, i.e., $A^* = A$.

(biii) Show that the matrix A is positive definite, i.e., (A is self-adjoint and)

$$X^t \, A \, \overline{X} > 0$$

for all nonzero column vector X.

END OF PAPER