NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION (2010–2011)

MA1521 Calculus for Computing

April 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** the questions.
3. Non-programmable calculators may be used. However, you should lay out systematically the various steps in your calculations.

Question 1 [10 marks]

Let $f(x,y) = 2x^3 - 12xy + 6y^2 - 18y - 12$.

(i) Find the coordinates of all critical points of f, and determine the nature of each critical point.

(ii) Find the directional derivative of f at the point $(1,1)$ in the direction of the **unit** vector $-\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

Question 2 [10 marks]

The function f is defined by

$$f(x) = \int_{x^2}^{x^3} \sqrt{7 + 2e^{3t-3}}\, dt, \qquad x > 0.$$

(i) Find the exact value of $f'(1)$.

(ii) Find the exact value of $\lim_{x \to 1} \frac{\sin((x-1)^2)}{(f(x))^2}$.

Question 3 [10 marks]

Find the Taylor series expansion for the function $f(x) = \frac{x^2 - 4x + 4}{2x^2 - 8x + 5}$ about $x = 2$.

Hence, determine the exact value of

(i) $f^{(10)}(2)$,

(ii) $g^{(11)}(2)$, where $g(x) = xf(x)$.

Question 4 [10 marks]

Use the method of variation of parameters to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = \frac{3x}{e^{3x}}.$$

Question 5 [10 marks]

Let a, b and c be positive real numbers and define

$$f(x, y, z) = x^a y^b z^c \quad \text{for } x \geq 0,\ y \geq 0,\ z \geq 0.$$

It is given that f attains its absolute maximum on the plane $x + y + z = 1$.

(i) Use the method of Lagrange multiplier to show that the absolute maximum of f is

$$\frac{a^a\, b^b\, c^c}{(a+b+c)^{a+b+c}}.$$

(ii) Deduce from (i) that for any positive real numbers p, q and r,

$$\left(\frac{p}{a}\right)^a \left(\frac{q}{b}\right)^b \left(\frac{r}{c}\right)^c \leq \left(\frac{p+q+r}{a+b+c}\right)^{a+b+c}.$$

Question 6 [15 marks]

Test the following series for convergence. Justify your answer.

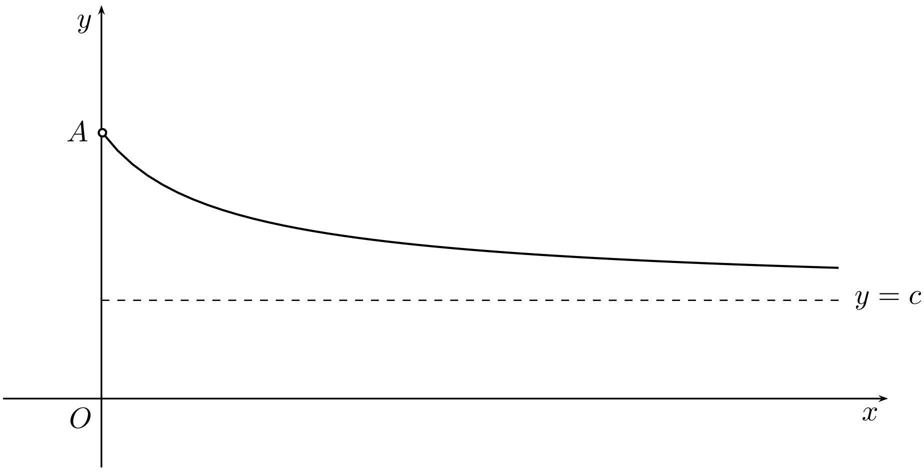
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+5)!}{e^n (n!)^2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2\sin(n^n) + 3\cos(n^{2n})}{4^{n/5}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\cos\frac{2}{n} - \cos\frac{4}{n}\right)$.

Question 7 [15 marks]

The diagram below shows the graph of the function $f(x) = (1+x)^{1/x}$, $x > 0$, which meets the y-axis at the point A and approaches the line $y = c$ as x tends to infinity.



(i) Find the value of c.

(ii) Find the coordinates of the point A.

A function g is said to have a fixed point at α if $g(\alpha) = \alpha$.

(iii) Show that the fixed point, α of the above function satisfies the equation

$$\alpha \ln \alpha - \ln(1+\alpha) = 0.$$

(iv) Find the integer n for which $1 + \frac{n}{10} < \alpha < 1 + \frac{n+1}{10}$.

(v) Apply the Newton-Raphson method to find α to three significant figures.

Determine whether your answer is greater or smaller than the actual value.

Question 8 [20 marks]

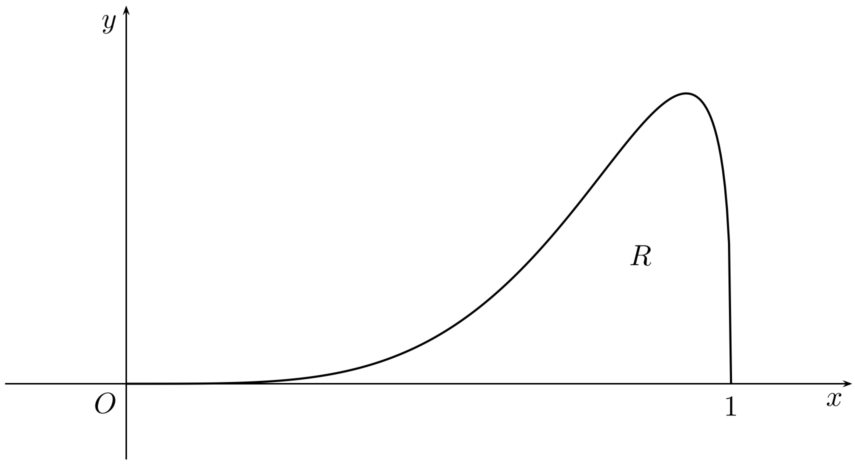
Let $I_n = \displaystyle\int_0^1 x^n(1-x^2)^{1/3}\,dx$ for $n \geq 0$.

(i) Find the exact value of I_1.

(ii) Use the trapezoidal rule with five ordinates to obtain an approximation to the integral I_0. Give your answer to four decimal places.

(iii) Prove that for all $n \geq 2$,

$$I_n = \frac{3n-3}{3n+5} I_{n-2}.$$



The above diagram shows the curve $y = x^4(1-x^2)^{1/3}$, $0 \leq x \leq 1$. The region R is bounded by the curve and the x-axis.

(iv) The region R is rotated completely about the y-axis. Use the reduction formula in (iii) to find the exact volume of the solid generated.

(v) Use the results in (ii) and (iii) to find an approximation for the area of the region R. Give your answer to three decimal places.

End of Paper