NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2010-2011

MA1104 Multivariable Calculus

April 2011 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-sized two-sided help sheets into the examination room.

2. This examination paper consists of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question; however, the marks for each part of the question are not necessarily the same.

4. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Answer **ALL** *questions.*

Question 1 (20 marks)

(a) Suppose

$$w = \ln(x + y), \quad x = e^t, \quad y = e^{-t}.$$

Using an appropriate Chain Rule, evaluate $\frac{d^2w}{dt^2}$ at $t = 0$.

(b) Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

(i) Do the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist? Compute them if they do.

(ii) Is the function $f_x(x,y)$ continuous at $(0,0)$? Justify your answer.

(iii) Is the function $f(x,y)$ differentiable at $(0,0)$? Justify your answer.

Question 2 (17 marks)

(a) The temperature (in degree Celcius) at a point (x,y) on a metal plate is

$$T(x,y) = 100 - 6xy - 5y^2.$$

An insect is crawling continuously along the ellipse $x^2 + 2y^2 = 6$ in the anti-clockwise direction on the metal plate. Suppose x and y are measured in meters. What is the rate of change of the temperature (degree Celcius per meter) when the insect is at the point $(2,1)$?

(b) Find the point on the paraboloid $z = \frac{x^2}{4} + \frac{y^2}{25}$ that is closest to the point $(3,0,0)$.

Question 3 (18 marks)

(a) Rewrite the following iterated integral using the order $dy\,dx\,dz$.

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} dz\,dy\,dx.$$

(b) Find the volume of the solid E bounded by the hyperboloid $x^2 + z^2 - y^2 = 2$ and the planes $y = 1$ and $y = 7$.

Question 4 (10 marks)

Let R be the region on the xy-plane bounded by the curves $y = \sqrt{x}$, $y = \sqrt{2x}$, $y = \frac{x^2}{3}$ and $y = \frac{x^2}{4}$.

Use the change of variables $x = u^{1/3}v^{2/3}$ and $y = u^{2/3}v^{1/3}$ to find the area of the region R.

Question 5 (20 marks)

(a) Consider the following force field defined on $\mathbb{R}^3$:

$$\mathbf{F}(x, y, z) = \left\langle y^2 \cos x + z^3, 2y \sin x - 4, 3xz^2 + 2 \right\rangle.$$

(i) Determine whether $\mathbf{F}(x, y, z)$ is a conservative vector field. Justify your answer.

(ii) Evaluate the work done by $\mathbf{F}(x, y, z)$ along the the straight line from $(0, 1, -1)$ to (π, π, π).

(b) Suppose $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$. Evaluate

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 5^2$ above the plane $z = 5/\sqrt{2}$ and is upwardly oriented.

Give your answer in three significant figures.

Question 6 (15 marks)

Consider the vector field

$$\mathbf{F}(x, y, z) = -\frac{z}{y}\ \mathbf{i} + y \sin y\ \mathbf{j} + z^2\ \mathbf{k}.$$

Let C_1 denote the closed curve which is the intersection of the cylinder $x^2 + z^2 = 1$ and the surface $y = 10 + x^2 + 3z^2$.

Let C_2 denote the closed curve which is the intersection of the cylinder $x^2 + z^2 = 1$ and the surface $x + y = 2$.

Suppose both C_1 and C_2 are oriented anti-clockwise when viewed from the positive y axis.

Is it true that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \quad ?$$

Justify your answer.

END OF PAPER